International Journal of Fracture **123:** 163–175, 2003.

A fracture criterion for sharp V-notched samples

F.J. GÓMEZ and M. ELICES*
Departamento de Ciencia de Materiales, Universidad Politécnica de Madrid, E.T.S. Ingenieros de Caminos, 28040 Madrid, Spain
*(*Author for correspondence: Fax: +34-9154537845; E-mail: melices@mater.upm.es)*

Received 30 October 2002; accepted in revised form 23 September 2003

Abstract. The objective of this paper is to show the advantages of the cohesive crack model for predicting fracture of V-notched components. Critical values of the generalized stress intensity factor can be obtained from the knowledge of the material softening function and the elastic parameters, avoiding a cumbersome experimental work. The results were checked successfully against experimental ones, from other authors, in different materials: steel, aluminium, PMMA and PVC. A non dimensional formulation of the fracture criterion for sharp V-notched components was obtained and a simple approximate expression derived for easy application.

Key words: cohesive crack, duraliminum, fracture criterion, fracture mechanics, PMMA, PVC, steel, V-notch.

1. Introduction

Notches in structural components give rise to localized stress concentrations which, in brittle materials, may generate a crack leading to catastrophic failure or to a shortening of the assessed structural life.

The importance of this problem has stimulated analysis and computations of the *strain* and *stress fields* in the V-notch tip region. To appreciate the work done in this area it suffices to look at recent papers by Strandberg (1999), Lazzarin (Filippi et al., 2002; Lazzarin et al., 2001), Seweryn (Seweryn and Lukaszewicz, 2002), Guo (Li and Guo, 2001; Guo, 2002) or Chen and Ushijima (2000) – to quote only a few – and references therein.

Fracture criteria for elements with V-notches is a different matter. A realistic modelling of the damage around the V-notch tip has proven very difficult and is strongly dependent on the microstructural aspects of each material. Therefore, fracture criteria are based on critical values of some macroscopic stress, strain or displacement fields, on non-local averaged parameters, or on educated assumptions (such as minimizing strain energy density). Good recent reviews of this subject have been published by Strandberg (2002) and Seweryn and Lukaszewicz (2002), where criteria based on strain energy release, strain energy density, critical stress at a certain distance and critical mean functions are considered.

When dealing with brittle, or quasi brittle materials – where linear elasticity can be applied – the stress and displacement fields near the tip of sharp V-notches can be characterized by the generalized stress intensity factor, K^V, which is a function of the V-notch angle. By analogy with cracked specimens, a fracture criterion based on critical values of the generalized stress intensity factors can be stated, i.e., a crack will propagate from the tip of a notch when the actual value of the generalized stress intensity factor reaches a critical value (Gradin, 1982; Carpinteri, 1987; Knesl, 1991; Seweryn, 1994; Dunn et al., 1997). The critical value of the generalized stress intensity factor K_C^V, of each V-notch angle, has to be obtained through experimental measurements on very sharp notched samples, frequently an involved procedure.

The purpose of this paper is to investigate the suitability of this fracture criterion for a wide range of brittle, or quasi brittle materials, whose fracture behaviour can be adequately modelled using the *cohesive crack* concept (Elices et al., 2002). It will be shown for these materials that:

- Critical values of the generalized stress intensity factor, K_C^V, can be obtained with little computational effort, from the knowledge of elastic parameters and the softening function.
- Critical values of the generalized stress intensity factor, K_C^V, computed using the cohesive crack concept are in good agreement with experimental results – from other authors – for different types of materials: steel, aluminum, PMMA and PVC.
- A non-dimensional formulation of the fracture criterion for sharp V-notched samples – the same for the whole family of materials with the same softening function– was obtained and a simple approximate expression derived for easy application.

2. Computation of critical values of generalized stress intensity factors

As already mentioned, critical values of the generalized stress intensity factors are obtained experimentally. Tests are performed on sharp V-notched specimens and the critical loads are measured. From these results the generalized stress intensity factors are computed. These notches have to be very sharp (usually the tip notch radius is less than 10 microns). Different angles, in the range from 0° to 180°, have to be tested and quite often different geometries are advisable when scatter appears. All these procedures may become cumbersome.

When cracking can be modelled using the *cohesive crack* concept, critical stress intensity factors can be computed, provided the softening function is known. Clearly, the parameters characterizing the softening function have to be measured. In the simplest approach, the specific fracture energy (or the fracture toughness) and the cohesive tensile stress are the needed parameters. Although in some circumstances these measurements may become involved, the problem is then limited to a computational one for each geometry and notch angle.

The V-notch problem for linear elastic materials was first analized by Williams (1952) by means of an eigenfunction series expansion and later by England (1971) using complex potentials. More recently, solutions valid for cracked and notched components have been published by Seweryn and Molski (1996) by Lazzarin and Tovo (1996) and by Filippi et al. (2002). These last approaches were based on complex stress functions, according to Muskhelishvili's method. It was shown that for mode I loading, the first term is always singular, and that the stress distribution at the notch tip can be described as

$$\sigma_{ij}(r,\theta) = \frac{K_I^V}{\sqrt{2\pi}r^{1-\lambda}} f_{ij}(\theta,\lambda), \tag{1}$$

where r and θ are the polar coordinates (as shown in Figure 1), K_I^V is the generalized stress intensity factor, λ is a function of the notch angle, defined below, and f_{ij} a function of the polar angle and, implicitly, of the notch angle.

The parameter λ characterizes the strength of the singularity and is the root of the equation,

$$\sin(\lambda\beta) + \lambda\sin(\beta) = 0 \tag{2}$$

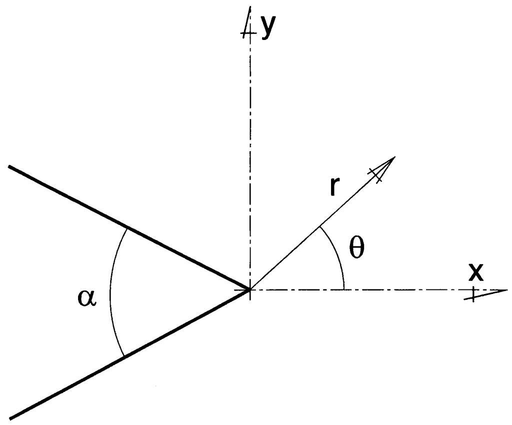

Figure 1. Coordinate system and symbols used for V-notches.

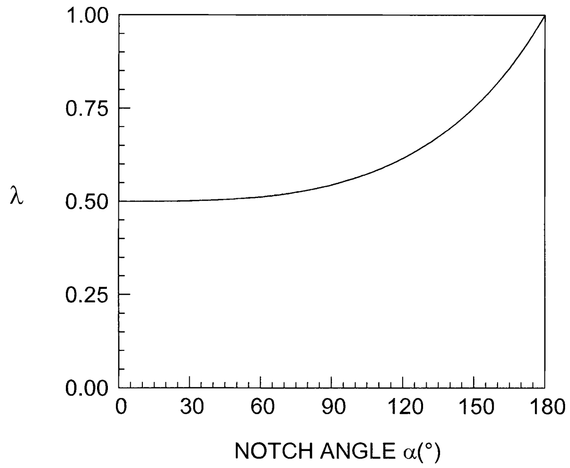

Figure 2. Values of λ as a function of the V-notch angle.

in which $\beta = 2\pi - \alpha$. Figure 2 shows the values of λ as a function of the notch angle (α).

When the stress distribution is known, the generalized stress intensity factor can be computed from the equation,

$$K_I^V = \sqrt{2\pi} \lim_{r \to 0^+} [r^{1-\lambda} \sigma_\theta(r, 0)] \tag{3}$$

which may be considered a definition of K_I^V (Gross and Mendelson, 1972).

In the limit, when $\alpha \to 0°$ and $\lambda \to 0.5$, Equation (1) reduces to the well-known crack tip stress field, and K_I^V coincides with the standard stress intensity factor K_I of fracture mechanics.

The generalized stress intensity factor K_I^V for sharp V-notches can also be computed using a path independent integral, according to Carpenter (1984), Atkinson et al. (1988) and Strandberg (1999). This is the procedure chosen for this paper. The integral can be written as

$$\int_C (\sigma_{ij}\hat{u}_i - \hat{\sigma}_{ij}u_i)n_j ds = K_I^V F(\alpha, E, \nu), \tag{4}$$

where σ_{ij} and u_i are computed stresses and displacements, $\hat{\sigma}_{ij}$ and $\hat{u}_i$ are auxiliary fields associated with – λ (see Atkinson et al., 1988), n_j are the components of the normal vector

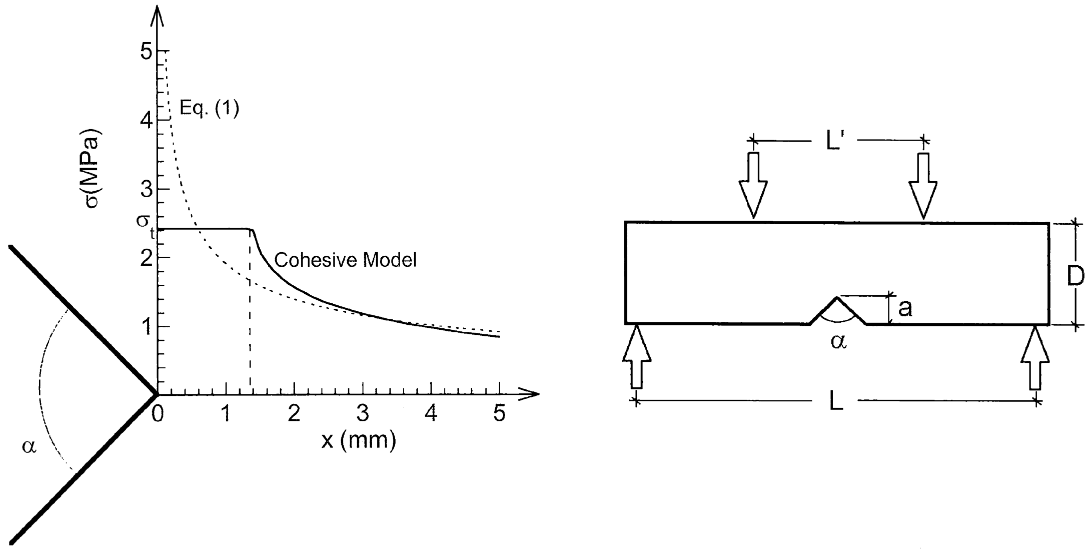

Figure 3. Stress distributions near the tip of a V-notch beam according to linear elasticity and the cohesive crack model.

directed outwards from the integration path C which encloses the V-notch, and E and ν are respectively the Young modulus and the Poisson's ratio of the material.

From Equation (4), provided that the height to width ratio of the specimen is large enough, one can write the following expression for the generalized stress intensity factor (Carpinteri, 1987; Strandberg, 1999).

$$K_I^V = \sigma_N D^{1-\lambda} f(\alpha, a/D), \qquad (5)$$

where σ_N is a nominal stress, D a geometrical dimension of the specimen (usually, specimen depth), a the notch depth, and $f(\alpha, a/D)$ a dimensionless function that depends on the considered geometry.

In this paper, we have chosen the following values for the nominal stress:

$$\sigma_N = P/BD \quad \text{for tension specimens,} \qquad (6a)$$

$$\sigma_N = 3PL/2BD^2 \quad \text{for three point bending specimens,} \qquad (6b)$$

$$\sigma_N = 3P(L-L')/2BD^2 \quad \text{for four point bending specimens,} \qquad (6c)$$

where P is the load and L, D and B are respectively the length, depth and thickness of the specimen, and L' is shown in Figure 3.

The critical generalized stress intensity factors (when using the cohesive crack model) were computed from Equation (5), by the following procedure:

Critical nominal stresses, σ_N, were derived from (6), where critical values of P were computed using the finite element method, performing the calculations with the commercial programme Abaqus, version 6.1 (Abaqus, 2000). The size of the element decreases near the tip of the notch, and a band of 100 special elements was placed in the ligament ahead of the notch, whose size was equal to that of the specimen divided by 11 200 (for a beam of depth $D = 28$ mm, the size of these elements was 2.5 microns). The special elements in the

Table 1. Shape factor for V-notched specimens for $a/D = 0.5$

Angle	TPB specimens	FPB specimens	SENT specimens	DENT specimens
0	1.767	1.145	3.539	0.9697
20				0.9747
30	1.775		3.567	
40				0.9981
60	1.857		3.755	1.049
80				1.146
90	2.118	1.410	4.296	
100				1.311
120	2.770		5.638	1.585
140		2.251		2.055
150	4.483		9.072	
155		2.874		
160				2.936

cohesive zone were of the bar type with a relation between the force transmitted among the nodes and their displacement given by the softening function. (For further details see Gómez et al., 2000; Planas et al., 1999). Bulk elements were conventional; plane strain and linear, with four integration points.

The dimensionless functions, $f(\alpha, a/D)$ can be extracted from the computations, previously mentioned, but in this paper they were taken from the literature. For example, Table 1 shows values of the shape factor $f(\alpha, 0.5)$ for the considered geometries: for three point bending (TPB) specimens, values were taken from Gross et al. (1972), except for the 150°, computed by the authors. For four point bending (FPB) specimens, the values are from Grenestedt et al. (1996). The values of single-edge notched tension (SENT) specimens are from Strandberg (1999) and those of double-edge notched tension (DENT) specimens are from Seweryn (1994).

In Figure 3, stresses according to the cohesive crack model are compared with the first term of the eigenfunction expansion Equation (1). This particular example corresponds to a notched beam loaded at four points with $\alpha = 90°$ and $a/D = 0.5$. Computations were performed for PVC samples.

3. Validation of the approach based on the cohesive model with experimental data

To check the suitability of this approach, critical values of generalized stress intensity factors computed from the *cohesive crack* model were compared with experimental values, obtained by other authors, with PMMA, PVC, steel and aluminium.

To this end, we looked into the literature for experimental results with V-shaped notched specimens with two requirements: brittle fracture and sharp notches.

Table 2. Characteristic values for PMMA

E (MPa)	σ_t (MPa)	G_{IC} (N m^{-1})	Reference
2300	124	393	Dunn et al. (1997)
3000	104.9	115.7	Seweryn (1999)

3.1. PMMA SAMPLES

Polymethyl-methacrylate (PMMA), an amorphous glassy polymer, is a relatively homogeneous isotropic medium that exhibits brittle fracture behaviour, even at room temperature, when tested with cracks or sharp notches. Dunn et al. (1997) performed a complete series of tests with three-point bending (TPB) specimens. Seweryn (1994) also performed tests using double-edge notched tensile (DENT) specimens of PMMA. The values of the critical stress intensity factors K_{IC}^V from these experiments are drawn in Figures 4a and 4b.

Values of computed K_{IC}^V, based on the cohesive crack model, are also shown in Figures 4a and 4b. The softening function assumed for the cohesive elements was the simplest one: a rectangular curve. The two parameters needed were the specific fracture energy, G_{IC}, and the cohesive tensile strength, σ_t. Outside the cracked zone, PMMA was modelled as linear elastic material. The values of G_{IC} and σ_t, shown in Table 2, are taken from the quoted references.

The agreement between experimental and predicted values of K_{IC}^V is very good. In Figure 4a, experimental results by the authors, from two different geometries (TPB and SENT), are included to remark that critical values of K_I^V are only a function of the V-notch angle and not of specimen geometry. (These results are drawn in Figure 4a because our PMMA parameters were more similar to Dunn's results than to those of Seweryn).

3.2. STEEL SAMPLES

Strandberg (2002) tested V-notched samples of annealed tool steel, AISI 01, at −50 °C. Single-edge notched tension and three-point bend specimens, with notch angles ranging from 0° to 140° were used, and the maximum loads recorded. From these values, critical generalized stress intensity factors were deduced and are plotted in Figure 5.

To reproduce numerically these experimental results with the cohesive model, the following inputs, according to Strandberg (2002) data, were used for the steel:

– The bulk was modelled as an elastoplastic material,

$$\epsilon = \sigma/E \qquad \text{if } \epsilon \leq \epsilon_y, \tag{7a}$$

$$\epsilon = \sigma/E + \epsilon_0[(\sigma/\sigma_y)^n] \qquad \text{if } \epsilon > \epsilon_y, \tag{7b}$$

  where ϵ_y is the strain at yielding, $\sigma_y = 501$ MPa, $E = 205$ GPa, $n = 6.6$ and $\epsilon_0 = 0.0027$.

– The cohesive behaviour was modelled with a rectangular softening function, with $\sigma_t = 1170$ MPa (from a tensile test where necking was considered) and with $G_{IC} = 12$ kN m^{-1} (computed from $G_{IC} = K_{IC}^2/E'$, where $K_{IC} = 52$ MPam$^{1/2}$, and E' is the generalized elasticity modulus for plane strain with Poisson coefficient $\nu = 0.3$).

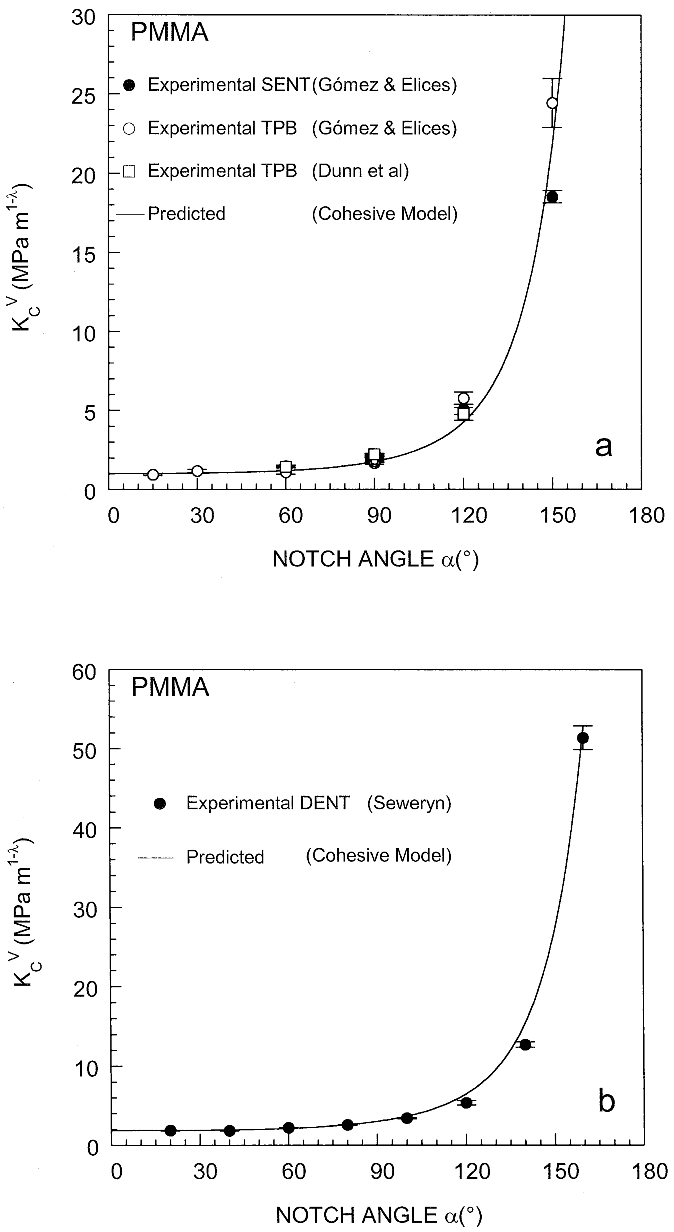

Figure 4. Experimental and predicted critical values of K_I^V in PMMA.

Numerical predictions are also shown in Figure 5. Agreement between computed and measured critical values is quite good in both types of samples (SENT and TPB), particularly as all the predictions were made from only two parameters (σ_t and G_{IC}) obtained from two independent tests.

3.3. PVC SAMPLES

Grenestedt et al. (1996) tested V-notched samples of expanded PVC foam of different densities: Divinycell H80, H100, H130 and H200, in which the numbers refer to densities in kg m^{-3}.

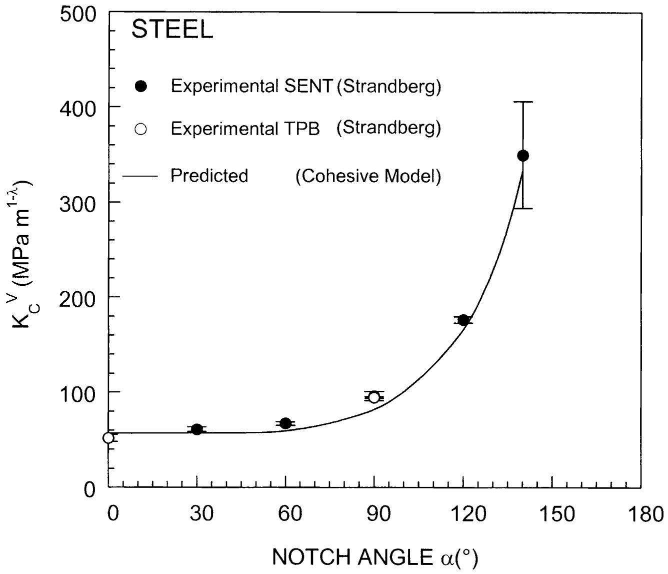

Figure 5. Experimental and predicted critical values of K_I^V in steel (at −50 °C).

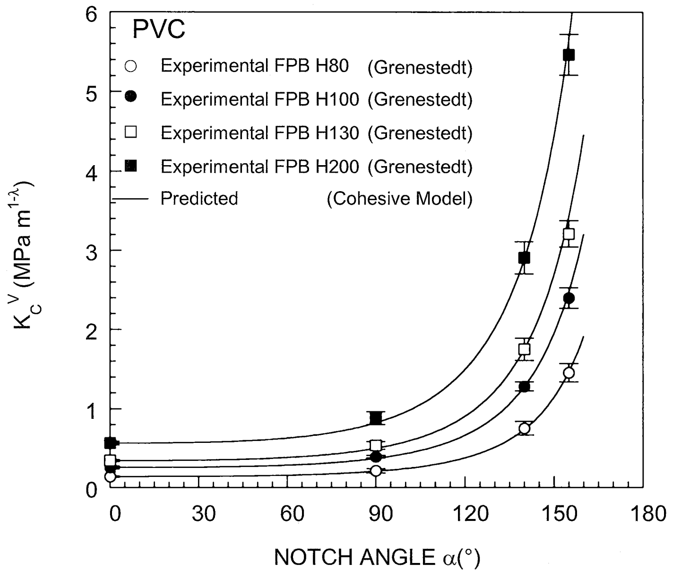

Figure 6. Experimental and predicted critical values of K_I^V in PVC.

Experiments were performed at room temperature on notched specimens loaded on four points (FPB), with notch angles of 0°, 90°, 140° and 155°. Maximum loads were recorded in all the experiments and the critical generalized stress intensity factors computed from them are plotted in Figure 6.

To compute K_{IC}^V for PVC using the cohesive model, the following inputs, according to Grenestedt et al. (1996), were used:

– The bulk was modelled as a linear elastic material whose Young modulus E is given in Table 3, according to the manufacturer Divinycell (2002).

Table 3. Mechanical properties of expanded PVC foam

Material	E (MPa)	K_{IC} (MPam$^{1/2}$)	G_{IC} (N m^{-1})	σ_t (MPa)
H80	85	0.14	210	2.51
H100	125	0.26	470	4.02
H130	175	0.34	610	5.70
H200	310	0.57	930	9.38

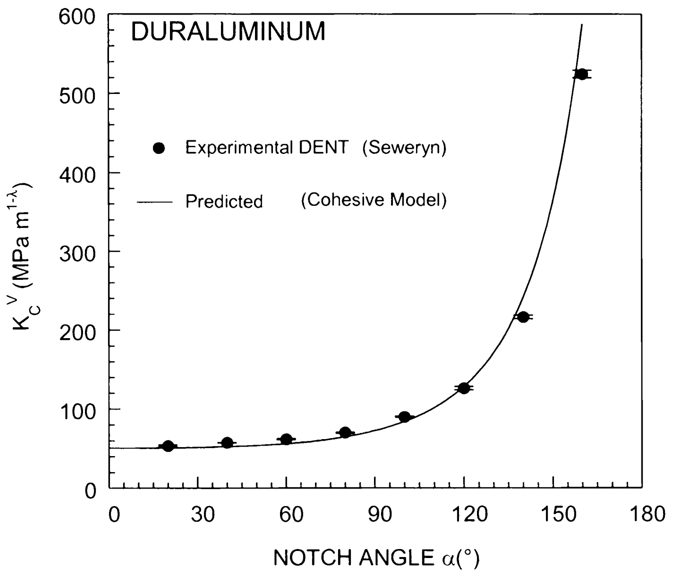

Figure 7. Experimental and predicted critical values of K_I^V in duraluminum.

– The cohesive behaviour was modelled using a rectangular softening function, with σ_t and G_{IC} values as shown in Table 3. Cohesive tensile strength (σ_t) and fracture toughness (K_{IC}) were measured by Grenestedt et al. (1996) through two independent tests. G_{IC} values were computed from $G_{IC} = K_{IC}^2/E'$, where E' is the generalized modulus and $\nu = 0.32$.

Numerical predictions together with experimental results are shown in Figure 6 for the four grades of expanded PVC foam. Once again, very good agreement was found between experimental critical values and numerical predictions based on the cohesive model.

3.4. Aluminium samples

Seweryn (1994) tested double-edge V-notched tension (DENT) samples made of duraluminium. The notch wedge angles were 20°, 40°, 60°, 80°, 100°, 120°, 140° and 160°. The wedge angles were found to be accurate to within 5°. The notch corner radii were smaller than 0.01 mm for all except the 20° notch samples. Critical values of the load under which a crack begins to propagate from the notch vertex were recorded and from these values critical generalized stress intensity factors were computed. These results are plotted in Figure 7.

To reproduce numerically the experimental results with the cohesive model, the following inputs, according to the Seweryn (1994) data, were used:

– The bulk was modelled as a linear elastic material, with Young modulus $E = 70$ GPa.
– The cohesive behaviour was modelled using a rectangular softening function, with $\sigma_t = 705.27$ MPa and $G_{IC} = 365$ N/m. G_{IC} was computed from $G_{IC} = K_{IC}^2/E'$, as always.

Numerical predictions are also shown in Figure 7. Once again, agreement between computed and measured critical values was quite good, adding further support to this procedure. Notice that although Seweryn realized that the fracture was not brittle, the good agreement between our computed values and the experimental results suggest that plasticity was well contained around the notch neighbourhood.

4. A non-dimensional formulation of the fracture criterion

The generalized stress intensity factor K_I^V, as defined in (2), is endowed with an odd dimensionality that depends on the notch angle α: When $\alpha = 0$, i.e., when dealing with a crack, $\lambda = 0.5$ (see Figure 2) and K_I^V becomes the well known stress intensity factor of Fracture Mechanics K_I^V, measured in Pa m$^{1/2}$. At the other end, when $\alpha = \pi$, $\lambda = 1.0$, and, according to (2), K_I^V is equal to $\sqrt{2\pi}\sigma_t$, and is measured in Pa. This prompted us to look for a nondimensional formulation by dividing K_I^V by $K_I r^{0.5-\lambda}$, where r is a suitable length.

Inside the frame of cohesive crack models, the natural choice of a length is the *characteristic length*:

$$l_{ch} = \frac{EG_{IC}}{\sigma_t^2}. \tag{8}$$

This length is an inverse measure of the brittleness of the material (the smaller the l_{ch} the more brittle the material). It is also related to the size of the fully developed fracture process zone (the size under peak load of the fracture process zone ahead of a semi-infinite crack in an infinite body).

To check the suitability of this nondimensional parameter, i.e.:

$$\frac{K_{IC}^V}{K_{IC} l_{ch}^{0.5-\lambda}} \equiv K_{IC}^{*,V}, \tag{9}$$

all the experimental results from PMMA (Dunn et al., 1997), steel (Strandberg, 2002), PVC (Grenestedt et al., 1996) and duraluminium (Seweryn 1994) are redrawn in Figure 8. All of them follow a common trend, in spite of the wide range of K_{IC} values considered: from 0.14 MPa m$^{1/2}$ of the PVC to 52 MPa m$^{1/2}$ of the steel.

Critical values of the nondimensional generalized stress intensity factor, $K_{IC}^{*,V}$, were also computed on the basis of the cohesive crack model with the assumption of a rectangular softening curve. Surprisingly, nondimensional curves for PVC, PMMA, aluminium and steel merged into one curve. This curve is also shown in Figure 8 and fits quite well the experimental data.

Generalization of these results involves certain considerations. One is that the simple softening curve – the rectangular one – is not necessarily the most accurate, although with the materials discussed in this paper it works quite well. Linear elastic behaviour was assumed and only sharp V-notches were considered. Rounded V-notches and the appearance of plasticity will shift the ideal predictions.

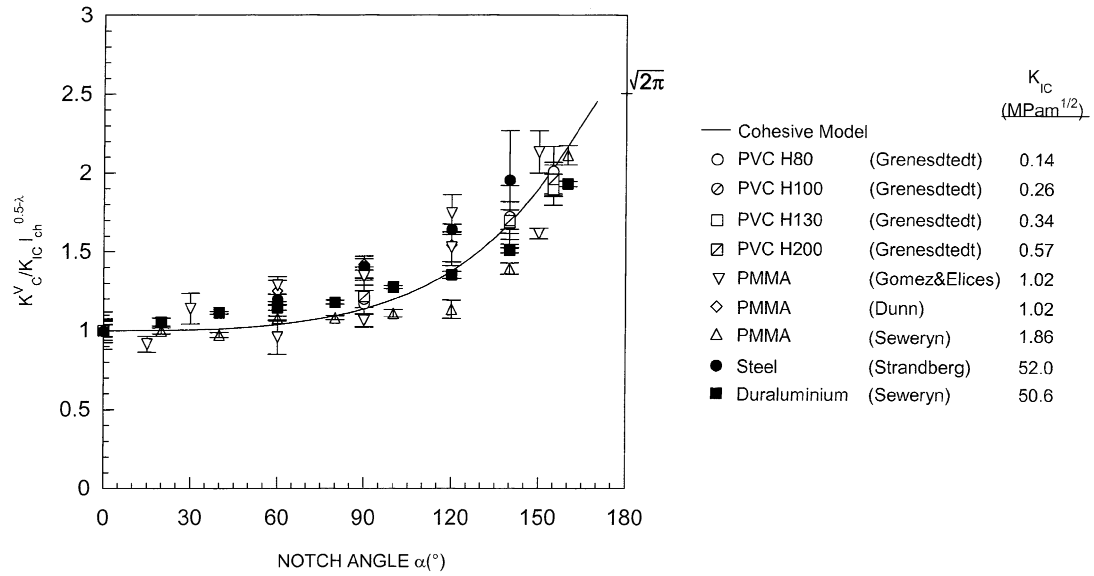

Figure 8. Critical values of the nondimensional generalized stress intensity factors: experimental results and theoretical predictions based on the cohesive model.

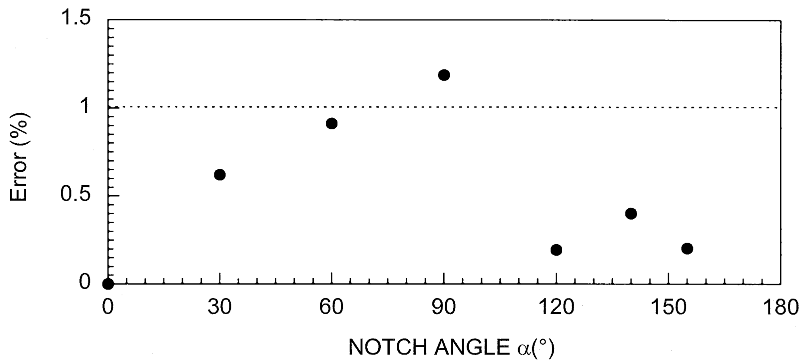

Figure 9. Accuracy of Equation (9) in comparison with numerical results.

To help in simple estimations of the brittle fracture loads in sharp V-notched components, an approximate expression for the nondimensional critical generalized stress intensity factor – in the range $0° \leq \alpha \leq 160°$ – can be written as:

$$K_{IC}^{*,V} = 1 + 0.038393\alpha^2 - 0.027857\alpha^3 + 0.024207\alpha^4. \quad (10)$$

The accuracy of this formula as compared to the computed one, based on the cohesive model, is shown in Figure 9. The error, computed as ((Equation (9)-computed)/computed, is less than 1% in almost all the angles considered. Nonetheless, the precautionary remarks on its applicability, already mentioned, should be taken into account.

Acknowledgements

The authors are grateful for useful discussions with Prof. Jaime Planas and they acknowledge the support of the Spanish Ministry of Ciencia and Tecnología under grants MAT 2000-1334 and 2FD97-1513-C03.01.

References

Abaqus. (2000). Abaqus/Standard user's manual, version 6.1. Hibbitt, Karlsson and Sorensen, Inc.

Atkinson, C., Bastero, J.M. and Martínez-Esnaola, J.M. (1998). Stress analysis in sharp angular notches using auxiliary fields. *Engineering Fracture Mechanics* **31**, 637–646.

Carpenter, W.C. (1984). A collocation procedure for determining fracture mechanics parameters at a corner. *International Journal of Fracture* **24**, 255–266.

Carpinteri, A. (1987). Stress-singularity and generalized fracture toughness at the vertex of re-entrant corners. *Engineering Fracture Mechanics* **26**, 143–155.

Chen, D.H. (1995). Stress intensity factors for V-notched strip under tension or in plane bending. *International Journal of Fracture* **70**, 81–97.

Chen, D.H. and Ushijima, K. (2000). Plastic stress singularity near the tip of a V-notch. *International Journal of Fracture* **106**, 117–134.

Dunn, M.L., Suwito, W. and Cunningham, S.J. (1997). Fracture initiation at sharp notches: Correlation using critical stress intensities. *International Journal of Solids and Structures* **34**, 3873–3883.

Elices, M., Guinea, G.V., Gómez, J. and Planas, J. (2002). The cohesive zone model. Advantages, limitations and challenges. *Engineering Fracture Mechanics* **69**, 137–163.

England, A.H. (1971). On stress singularities in linear elasticity. *International Journal Engineering Science* **9**, 571–585.

Filippi, S., Lazzarin, P. and Tovo, R. (2002). Developments of some explicit formulas useful to describe elastic stress fields ahead of notches in plates. *International Journal of Solids and Structures* **39**, 4543–4565.

Gómez, F.J., Elices, M. and Valiente, A. (2000). Cracking in PMMA containing U-shaped notches. *Fatigue and Fracture of Engineering Materials Structures* **23**, 795–803.

Gradin, P.A. (1982). A fracture criterion for edge-bonded bi-material bodies. *Journal Composites Material* **16**, 448–456.

Grenestedt, J.L., Hallström, S. and Kuttenkeuler, J. (1996). On cracks emanating from wedges in expanded PVC foam. *Engineering Fracture Mechanics* **54**, 445–456.

Gross, B. and Mendelson, A. (1972). Plane elastostatic analysis of V-notched plates. *International Journal of Fracture* **8**, 267–276.

Guo, W. (2002). Theoretical investigation of elastoplastic notch fields under triaxial stress constraint. *International Journal of Fracture* **115**, 233–249.

Knésl, Z. (1991). A criterion of V-notch stability. *International Journal of Fracture* **48**, R79–R83.

Lazzarin, P. and Tovo, R. (1996). A unified approach to the evaluation of linear elastic stress fields in the neighbourhood of cracks and notches. *International Journal of Fracture* **78**, 3–19.

Lazzarin, P., Zambardi, R. and Livieri, P. (2001). Plastic notch stress intensity factors for large V-shaped notches under mixed load conditions. *International Journal of Fracture* **107**, 361–377.

Li, Z. and Guo, W. (2001). Three-dimensional elastic stress fields ahead of blunt V-notches in finite thickness plates. *International Journal of Fracture* **107**, 53–71.

Pildey, W.D. (1997). *Peterson's Stress Concentration Factors*. John Wiley and Sons Inc., New York.

Planas J., Guinea G.V., Elices M. (1999). Integral equation method for modeling cracking in concrete. *Computational Fracture in Concrete Technology* (Edited by M.H. Aliabadi and A. Carpinteri), Chapter 4, WIT Press, Boston, Southampton, 103–132.

Seweryn, A. (1994). Brittle fracture criterion for structures with sharp notches. *Engineering Fracture Mechanics* **47**, 673–681.

Seweryn, A. and Lukaszewicz, A. (2002). Verification of brittle fracture criteria for elements with V-shaped notches. *Engineering Fracture Mechanics* **69**, 1487–1510.

Seweryn, A. and Molski, K. (1996). Elastic stress singularities and corresponding generalized stress intensity factors for angular corners under various boundary conditions. *Engineering Fracture Mechanics* **55**, 529–556.

Strandberg, M. (1999). A numerical study of the elastic stress field arising from sharp and blunt V-notches in a SENT-specimen. *International Journal of Fracture* **100**, 329–342.

Strandberg, M. (2002). Fracture at V-notches with contained. *Engineering Fracture Mechanics* **69**, 403–415.

Williams, M.L. (1952). Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *Journal of Applied Mechanics* **19**, 526–528.