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About an example of a Banach space not weakly K-analytic

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Abstract. Cascales, Kąkol, Saxon proved [3] that in a large class $\mathfrak{G}$ of locally convex spaces (lcs) (containing (LM)-spaces and (DF)-spaces) for a lcs $E \in \mathfrak{G}$ the weak topology $\sigma(E, E')$ of E has countable tightness iff its weak dual $(E', \sigma(E', E))$ is K-analytic. Applying examples of Pol [9] (and Kunen [12]) one gets that there exist Banach spaces $C(X)$ over a compact scattered space X such that $C(X)$ is not weakly K-analytic (even not weakly K-countably determined under (CH)) but the weak dual of $C(X)$ has countable tightness. This provides also an example showing that (gDF)-spaces need not be in class $\mathfrak{G}$.

Sobre un ejemplo de un espacio de Banach no débilmente K-analítico

Resumen. Cascales, Kąkol, Saxon probaron en [3] que en la amplia clase $\mathfrak{G}$ de espacios localmente convexos, que contienen a los espacios (LM) y a los espacios (DF), se tiene que un espacio $E \in \mathfrak{G}$ verifica que su topología débil $\sigma(E, E')$ tiene *tightness* numerable si y sólo si su dual débil $(E', \sigma(E', E))$ es K-analitico. Aplicando ejemplos de Pol [9] (y de Kunen [12]) se obtiene que existen compactos diseminados (*scattered*) X tales que el espacio de Banach $C(X)$ no es débilmente K-analitico (y ni siquiera es débilmente K-numerablemente determinado, si se admite la hipótesis del continuo), pero el dual débil de $C(X)$ tiene tightness numerable, lo que proporciona de exemplo de espacio (DF) generalizado (espacio (gDF)) que no pertenece a la clase $\mathfrak{G}$.

1 Introduction

Many concrete locally convex spaces (lcs) E are *weakly realcompact*, i.e. the weak topology $\sigma(E, E')$ is realcompact. For example any lcs E whose strong dual is metrizable and separable has this property. This follows from (*): *If* (E, E') *is a dual pair and* $\sigma(E, E')$ *has countable tightness* (i.e. for every $A \subset E$ and $x \in \overline{A}$ there exists countable $B \subset A$ such that $x \in \overline{B}$), *then* $\sigma(E', E)$ *is realcompact*. Indeed, by Corson [4], see also [17, p.137], it is enough to show that every linear functional f on E which is $\sigma(E, E')$-continuous on each $\sigma(E, E')$-closed separable vector subspace is continuous. Note that $K := \overline{f^{-1}(0)}$ is closed in $\sigma(E, E')$: If $y \in \overline{K}$, then there is countable $D \subset K$ with $y \in \overline{D}$. By assumption $f|\overline{\mathrm{lin}(D)}$ is $\sigma(E, E')$-continuous, so $f(y) \in \overline{f(\mathrm{lin}(D))} \subset \overline{\mathrm{lin}\, f(K)} = \{0\}$. Hence $y \in K$, so f is continuous.

The following is much less evident: *Every Banach space with Corson property* (C) *is weakly realcompact*, [4, 11]. It turns out that in a class $\mathfrak{G}$ of lcs (containing all (LM)-spaces and (DF)-spaces) the converse to (*) holds even in a stronger form [3].

Proposition 1 *If* $E \in \mathfrak{G}$, *then* $E_\sigma := (E, \sigma(E, E'))$ *has countable tightness iff* $E'_\sigma := (E', \sigma(E', E))$ *is* K-*analytic.*

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Is it true that a Banach space E is weakly Lindelöf if E'_σ has countable tightness? If $\sigma(E,E')$ is Lindelöf, is the unit ball in E' of countable tightness in $\sigma(E',E)$? This problem is strictly related with Banach spaces satisfying the Corson property (C), i.e. every collection of closed convex subsets of E with empty intersection contains a countable subcollection with empty intersection [4]. Every Banach space E which is $\sigma(E,E')$-Lindelöf has property (C); the converse fails, see [11] and references.

Pol [11] proved that a *Banach space* E *has property* (C) *iff* E'_σ *has property* (C'), i.e. for every $A\subset E'$ and $f\in\overline{A}$ (the closure in $\sigma(E',E)$) there exists a countable set $B\subset A$ with $f\in\overline{\operatorname{conv} B}$. Still it is unknown if property (C') can be replaced by *countable tightness* of the unit ball in $\sigma(E',E)$, see [11, 5]. The aim of this short note is to prove Proposition 2 yielding Example 1 and Example 2 stated in Abstract, which answer also a question of Professor P. Domański (personal communication). This applies to show that (gDF)-spaces need not to be in class $\mathfrak{G}$.

A lcs E belongs to class $\mathfrak{G}$ if there is a family $(A_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ of subsets in E' covering E' such that $A_\alpha\subset A_\beta$ for $\alpha\leqslant\beta$ and in each A_α sequences are equicontinuous [2]. All (LM)-spaces (hence metrizable lcs) and dual metric spaces (hence (DF)-spaces) belong to $\mathfrak{G}$ by Examples from [2].

A topological space X is called *Lindelöf* Σ (called also K-countable determined) if there is an upper semi-continuous (usco) map from a nonempty subset $\Sigma\subset\mathbb{N}^\mathbb{N}$ with compact values in X whose union is X, where the set of integers $\mathbb{N}$ is discrete and $\mathbb{N}^\mathbb{N}$ is endowed with the product topology [8, 1, 6]. If the same holds for $\Sigma=\mathbb{N}^\mathbb{N}$, then X is called K*-analytic*.

2 One Proposition and Example

For a Tichonov space X by $C_p(X)$ we denote the space of continuous realvalued maps on X endowed with the pointwise topology. In [5, Theorem 3.6] it was proved that if X is a compact zero-dimensional space and $C(X)$ is weakly Lindelöf, then the unit ball in the dual $M(X)$ of $C(X)$ has countable tightness in its weak dual topology, and if X is compact and scattered, then $C(X)$ has property (C) iff X has countable tightness [11, Corollary 1].

For compact scattered X we show also the following simple fact which will be used below.

Proposition 2 *Let X be a scattered compact space such that $C(X)$ is weakly Lindelöf. Then the weak dual of $C(X)$ has countable tightness.*

PROOF. By assumption also $C_p(X)$ is Lindelöf. Let τ_p and $\tau_\sigma:=\sigma(C(X),C(X)')$ be the topology of $C_p(X)$ and the weak topology of $C(X)$, respectively. Let B be the closed unit ball in $C(X)$. Since X is scattered, then $\tau_p|B=\tau_\sigma|B$ by [15, Corollary 19.7.7]. But then $\tau_p^n|B^n=\tau_\sigma^n|B^n$ for each $n\in\mathbb{N}$, where $B^n:=\prod_{1\le i\le n}B$ and τ_p^n, τ_σ^n denote the own product topologies on the product $C(X)^n:=\prod_{1\le i\le n}C(X)$. Since X as compact and scattered is zero-dimensional [1, Theorem IV.8.6] applies to deduce that $(C(X)^n,\tau_p^n)$ is Lindelöf for each $n\in\mathbb{N}$. But then B^n (closed in τ_p^n) is also Lindelöf. Hence B^n is Lindelöf in $(C(X)^n,\tau_\sigma^n)$ for each $n\in\mathbb{N}$. Since $C(X)^n=\bigcup_m mB^n$ and each mB^n is Lindelöf in τ_σ^n, then $(C(X)^n\tau_\sigma^n)$ is a Lindelöf space for each $n\in\mathbb{N}$. But [1, Theorem II.1.1] implies that $C_p((C(X),\tau_\sigma))$ has countable tightness. Consequently $(C(X)',\sigma(C(X)',C(X))\subset C_p(C(X),\tau_\sigma)$ has countable tightness. ■

Let $\Gamma\subset\Omega$ be the set of all non-limit ordinals and let $\Lambda=\Omega\backslash\Gamma$. Attach to each $\gamma\in\Lambda$ an increasing sequence $(s_\lambda(n))_n$ in Γ such that $\lim_{n\to\infty}s_\lambda(n)=\lambda$. Endow the set Ω with the following topology: The points from Γ are isolated and the basic neighbourhoods of $\gamma\in\Lambda$ are of the form $W_\gamma(n)=\{\gamma\}\cup\{\,s_\gamma(m):m\ge n\,\}$. Let $X_0=\Omega\cup\{\omega_1\}$ be the one-point compactification of the locally compact space Ω. Then X_0 is scattered. By [9] the Banach space $C(X_0)$ is weakly Lindelöf and by [16] it is not a weakly compactly generated (WCG) Banach space. Since every weakly K-analytic Banach space $C(X)$ over compact scattered X is (WCG) by [10], then $C(X_0)$ is not weakly K-analytic. Using Proposition 2 one gets an example showing that a variant of Proposition 1 fails in the following sense.

Example 1 *The weak dual of $C(X_0)$ has countable tightness but the weak topology of $C(X_0)$ is not K-analytic.*

Last example may suggest the following question: Let X be a compact scattered space such that $C(X)$ is a weakly Lindelöf space. Is then $C(X)$ a weakly Lindelöf Σ-space? The answer is negative:

Under Continuum Hypothesis (CH) Kunen provided and example, see [12, 9], of a scattered compact separable space X with cardinality $\aleph_1$ such that $C(X)$ is weakly Lindelöf. From Proposition 2 the weak dual of $C(X)$ has countable tightness. Note that the weak topology of $C(X)$ is not Lindelöf Σ. It is enough to show that $C_p(X)$ is not Lindelöf Σ. Since X is separable, it is enough to show that X does not have a countable network (or equivalently X is not cosmic, i.e. X is not a continuous image of a metric separable space) owing to [1, Corollary IV.9.9]. Assume that X is cosmic. Then also $C_p(X)$ is cosmic by [7, Corollary 4.1.3]. But then $C_p(X)$ would be separable yielding on X a weaker metric topology. It is however well-known that a metric compact scattered space is countable, a contradiction. So we have

Example 2 *Under (CH) there exists a compact scattered space X such that $C(X)$ is weakly Lindelöf but not weakly Lindelöf Σ and the weak dual of $C(X)$ has countable tightness.*

As we have mentioned any (DF)-space belongs to class $\mathfrak{G}$. A natural generalization of the concept of (DF)-spaces is the class of (gDF)-spaces. Following Ruess a lcs (E, ξ) is a (gDF)-space if it has a fundamental sequence of bounded sets $(S_n)_n$ and ξ is the finest locally convex topology on E of all agreeing with ξ on all sets S_n. If E is a Fréchet space, then the Mackey dual $(E', \mu(E', E))$ is a (gDF)-space, [13, 14]. Set $E := C(X_0)$, where $C(X_0)$ is as above. Assume that $(E', \mu(E', E)) \in \mathfrak{G}$. Since the weak dual of $C(X_0)$ has countable tightness by Proposition 2, then Proposition 1 implies that the weak topology of $C(X_0)$ is K-analytic, a contradiction. Therefore, although (gDF)-spaces seem to be "close" to (DF)-spaces, we have

Corollary 1 *There exist (gDF)-spaces which do not belong to class $\mathfrak{G}$.*

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