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A note on Fréchet-Urysohn locally convex spaces

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Abstract. Recently Cascales, Kąkol and Saxon showed that in a large class of locally convex spaces (so called class $\mathfrak{G}$) every Fréchet-Urysohn space is metrizable. Since there exist (under Martin's axiom) non-metrizable separable Fréchet-Urysohn spaces $C_p(X)$ and only metrizable spaces $C_p(X)$ belong to class $\mathfrak{G}$, we study another sufficient conditions for Fréchet-Urysohn locally convex spaces to be metrizable.

Una nota sobre espacios de Fréchet-Urysohn localmente convexos

Resumen. Recientemente Cascales, Kąkol y Saxon han probado que en una amplia clase de espacios localmente convexos (llamada clase $\mathfrak{G}$) los espacios con la propiedad de Fréchet-Urysohn son metrizables. Si se admite el axioma de Martin existen espacios $C_p(X)$ separables que tienen la propiedad de Fréchet-Urysohn y que no son metrizables. La metrizabilidad de los espacios $C_p(X)$ que pertenecen a la clase $\mathfrak{G}$ ha motivado el que se estudie en este artículo condiciones suficientes para que los espacios localmente convexos con la propiedad de Fréchet-Urysohn sean metrizables.

1 Introduction

One of the interesting and difficult problems (Malyhin 1978) concerning Fréchet-Urysohn groups asks if every separable Fréchet-Urysohn topological group is metrizable [18]; see also [20] and [21] for some counterexamples under various additional set-theoretic assumptions. The same question can be formulated in the class of locally convex spaces (lcs). Under Martin's axiom (MA) there exist non-metrizable analytic (hence separable) Fréchet-Urysohn spaces $C_p(X)$. On the other hand, *the Borel Conjecture* implies that separable and Fréchet-Urysohn spaces $C_p(X)$ are metrizable. In fact there exist many important classes of lcs for which the Fréchet-Urysohn property implies metrizablity. We showed in [3, Theorem 2] that (LM)-spaces, (DF)-spaces (in fact all spaces in class $\mathfrak{G}$) are metrizable if and only if they are Fréchet-Urysohn. In [22, Theorem 5. 7] Webb proved that only finite-dimensional Montel (DF)-spaces enjoy the Fréchet-Urysohn property. We extend this fact by noticing that every Fréchet-Urysohn hemicompact topological group is a Polish space. The aim of the rest part of the paper is to characterize metrizability of Fréchet-Urysohn lcs in terms of certain resolutions. First we prove that for a lcs X its strong dual F is metrizable if and only if F is Fréchet-Urysohn and X has a bornivorous bounded resolution. This applies to observe that the space of distributions $D'(\Omega)$ and the space $A(\Omega)$ of real analytic functions on an open set $\Omega \subset \mathbb{R}^{\mathbb{N}}$ are not Fréchet-Urysohn (although they have countable tightness by [3, Corollary 2. 4]). Nevertheless, there exist Fréchet-Urysohn non-metrizable lcs which admit a bornivorous bounded resolution, see Example 1. We show however that a Frćhet-Urysohn lcs is metrizable if and only if it admits a superresolution.

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2 Notations, Definitions and Elementary Facts

A Hausdorff topological space (space) X is said to have a compact *resolution* if X is covered by an ordered family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, that is, such that $A_\alpha \subset A_\beta$ for $\alpha \leq \beta$. Any K-analytic space has a compact resolution but the converse implication fails in general [19]. For a lcs X a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is called *bounded* if each set A_α is bounded in X. If additionally every bounded set in X is absorbed by some K_α, the resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is called *bornivorous*. For $\alpha = (n_k)_k \in \mathbb{N}^\mathbb{N}$ put $C_{n_1 n_2 \ldots n_k} := \bigcup \{K_\beta : \beta = (m_l)_l, m_j = n_j, j = 1, \ldots, k\}$. Clearly $K_\alpha \subset C_{n_1,\ldots,n_k}$ for each $k \in \mathbb{N}$. A lcs X is said to have a *superresolution* if for every finite tuple $(n_1, \ldots n_p)$ of positive integers and every bounded set Q in X there exists $\alpha = (m_k) \in \mathbb{N}^\mathbb{N}$ such that $m_j = n_j$ for $1 \leq j \leq p$ and K_α absorbs Q. This implies that for any finite tuple $(n_1, \ldots n_p)$ the sequence $(C_{n_1,\ldots,n_p,n})_n$ is bornivorous in X.

Clearly any metrizable lcs X admits a superresolution: For a countable basis $(U_k)_k$ of neighbourhoods of zero in X and $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ set $K_\alpha := \bigcap_k n_k U_k$. Then $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is as required.

Moreover, (DF)-spaces, regular (LM)-spaces *admit a bornivorous bounded resolution*. Indeed, let X be an (LM)-space and let $(X_n)_n$ be an increasing sequence of metrizable lcs whose inductive limit is X, see [15] for details. For each $n \in \mathbb{N}$ let $(U_k^n)_k$ be a countable basis of absolutely convex neihbourhoods of zero in X_n such that $U_k^n \subset U_k^{n+1}$ and $2U_{k+1}^n \subset U_k^n$ for each $k \in \mathbb{N}$. For each $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ set $K_\alpha := \bigcap_k n_k U_k^{n_1}$. Then $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a bounded resolution. In fact, $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ covers X and the set $\bigcap_k n_k U_k^{n_1}$ is bounded in X_{n_1}, hence in the limit space X. If additionally X is regular, i.e. for every bounded set B in X there exists $m_1 \in \mathbb{N}$ such that B is contained and bounded in X_{m_1}, then for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $B \subset \bigcap_k n_k U_k^{m_1}$. This yields a sequence $\alpha = (m_k) \in \mathbb{N}^\mathbb{N}$ such that $B \subset K_\alpha$. Any lcs admitting a fundamental sequence $(B_n)_n$ of bounded sets has a superresolution; put $K_\alpha := K_{n_1}$ for $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$.

A space X is *Fréchet-Urysohn* if for each set A in X and any $x \in \overline{A}$ there exists a sequence of elements of A converging to x.

A lcs X is *barrelled* (*quasibarrelled*) if every closed absolutely convex absorbing (bornivorous) subset of X is a neighbourhood of zero. A lcs X is called *Baire-like* [17] (*b-Baire-like* [16]) if for every increasing (and bornivorous) sequence $(A_n)_n$ of absolutely convex closed subsets of E there exists $n \in \mathbb{N}$ such that A_n is a neighbourhood of zero in E. Every metrizable (metrizable and barrelled) lcs is b-Bairelike (Baire-like) and every barrelled b-Baire-like space is Baire-like. Every Fréchet-Urysohn lcs is both b-Baire-like and bornological, [9]. By $C_p(X)$ and $C_c(X)$ we denote the spaces of real-valued continuous functions on a Tychonov space endowed with the topology of pointwise convergence and the compact-open topology, respectively.

3 Results and Remarks

Note that for an uncountable compact scattered space X the space $C_p(X)$ is Fréchet-Urysohn [1, Theorem III. 1. 2] non-metrizable and admits a bounded resolution. (**) *Is a Fréchet-Urysohn lcs metrizable if it admits a strongly bounded resolution*? Clearly every lcs with a fundamental sequence of bounded sets generates a bornivorous bounded resolution. In [3] we proved that every Fréchet-Urysohn lcs in class $\mathfrak{G}$ is metrizable, so every Féchet-Urysohn (DF)-space is normable. Nevertheless, we have the following

Example 1 *Let X be a Σ-product of $\mathbb{R}^I$, with uncountable I, formed by all sequences of countable support. Then X contains a Fréchet-Urysohn non-metrizable vector subspace having a bornivorous bounded resolution.*

PROOF. Clearly X is a Fréchet-Urysohn space, see [13]. Let G be the linear span of the compact set $B := [-1, 1]^I$. Set $E := F \cap G$ and denote $B \cap F$ by C. For $\alpha = (n_k)$ set $K_\alpha = n_1 C$. Note that $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a strongly bounded resolution in E. First observe that the family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is ordered, covers E and each K_α is a bounded set in E. Next, let P be a bounded set in E and let A be its closed absolutely convex cover in E. Then A is bounded in $\mathbb{R}^I$, hence relatively compact in $\mathbb{R}^I$, and since every cluster point of a countable set in X has countable support, A is countably compact in X. But since A is closed in E, we conclude that A is a countably compact subset of E. It is easy to see that A is a Banach disk, i.e. its linear span E_A endowed with Minkowski functional norm is a Banach space. Since $\{nC \cap E_A : n \in \mathbb{N}\}$ is a sequence of closed absolutely convex sets in E_A covering E_A and E_A is a Baire space, there is $m \in \mathbb{N}$ such that $A \subseteq mC$. So if $\beta = (n_k) \in \mathbb{N}^{\mathbb{N}}$ verifies that $n_1 = m$, then $A \subseteq K_\beta$, so that $P \subseteq K_\beta$. Therefore the family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a strongly bounded resolution in E as stated and E is non-metrizable since C is not metrizable. ■

In general we note the following

Proposition 1 *For a lcs X its strong dual $(X', \beta(X', X))$ is metrizable if and only if it is Fréchet-Urysohn and X admits a bornivorous bounded resolution.*

PROOF. If $(X', \beta(X', X))$ is metrizable, then it is Fréchet-Urysohn and X admits a bornivorous bounded resolution. Now assume that X admits a bornivorous bounded resolution $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Then the polars K_α° of K_α in the topological dual X' of X form a base of neighbourhoods of zero for the strong topology $\beta(X', X)$. But the polars $K_\alpha^{\circ\circ}$ in X'' compose a resolution consisting of equicontinuous sets covering X''. This shows that the space $(X', \beta(X', X))$ belongs to class $\mathfrak{G}$ (in sense of Cascales and Orihuela of [2]). If $(X', \beta(X', X))$ is Fréchet-Urysohn, then [3, Theorem 2] yields the metrizability of $(X', \beta(X', X))$. ■

Corollary 1 *The spaces $D'(\Omega)$ of distributions and $A(\Omega)$ of real analytic functions for an open set $\Omega \subset \mathbb{R}^{\mathbb{N}}$ are not Fréchret-Urysohn.*

PROOF. Since $D'(\Omega)$ is non-metrizable (quasibarrelled) and is the strong dual of a complete (LF)-space $D(\Omega)$ of the test functions, we apply Proposition 1. The same argument can be used to the space $A(\Omega)$ via [5, Theorem 1. 7 and Proposition 1. 7]. ■

Although (**) has a negative answer we note the following result for a large class of lcs. We need the following somewhat technical

Proposition 2 *A b-Baire-like space X is metrizable if and only if X admits a superresolution*

PROOF. Let $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a superresolution for X. We may assume that all sets K_α are absolutely convex. Then the sets $C_{n_1 n_2 \ldots n_k}$ (defined above) are also absolutely convex. First observe that *for every $\alpha = (n_k) \in N^{\mathbb{N}}$ and every neighbourhood of zero U in X there exists $k \in N$ such that $C_{n_1, n_2, \ldots, n_k} \subset 2^k U$.*

Indeed, otherwise there exists a neighbourhood of zero U in X such that for every $k \in \mathbb{N}$ there exists $x_k \in C_{n_1, n_2, \ldots, n_k}$ such that $x_k \notin 2^k U$. Since $x_k \in C_{n_1, n_2, \ldots, n_k}$ for every $k \in \mathbb{N}$, there exists $\beta_k = (m_n^k)_n \in \mathbb{N}^{\mathbb{N}}$ such that $x_k \in K_{\beta_k}$, $n_j = m_j^k$, $j = 1, 2, \ldots, k$. Set $a_n = \max\left\{m_n^k : k \in \mathbb{N}\right\}$ for $n \in \mathbb{N}$ and $\gamma = (a_n)_n$. Since $\beta_k \leq \gamma$ for every $k \in \mathbb{N}$, then $K_{\beta_k} \subset K_\gamma$, so $x_k \in K_\gamma$ for all $k \in \mathbb{N}$. Therefore $2^{-k} x_k \to 0$ which provides a contradiction. The claim is proved.

Let Y be the completion of X. It is clear that Y is Baire-like. Next, we show that there exists $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ such that $\overline{C_{n_1, n_2, \ldots, n_k}}$ is a neighbourhood of zero in X for each $k \in \mathbb{N}$. Assume that does not exist $n_1 \in \mathbb{N}$ such that $\overline{C_{n_1}}$ is a neighbourhood of zero. Since (by assumption) the sequence $(nC_n)_n$ is bornivorous in X and X is quasibarrelled, we apply [15, 8.2.27] to deduce that $Y = \overline{X} = \overline{\bigcup_n nC_n} \subset (1+\epsilon) \bigcup_n n\overline{C_n}$. But Y is a Baire-like space, so there exists $n_1 \in \mathbb{N}$ such that $\overline{C_{n_1}}$ is a neighbourhood of zero in Y.

Assume that for a finite tuple $(n_1, \ldots n_p)$ of positive integers the set $\overline{C_{n_1, \ldots, n_k}}$ is a neighbourhood of zero for each $1 \leq k \leq p$. Since, by assumption, the sequence $(nC_{n_1, \ldots, n_p, n})_n$ is bornivorous in X, and consequently $X = \bigcup_n nC_{n_1, \ldots, n_p, n}$, we apply the same argument as above to get an integer $n_{p+1} \in \mathbb{N}$ such that $\overline{C_{n_1, \ldots, n_p, n_{p+1}}}$ is a neighbourhood of zero, which completes the inductive step. This fact combined with the claim provides a countable basis $(2^{-k}\overline{C_{n_1, \ldots, n_k}})_k$ of neighbourhoods of zero, so X is metrizable. ■

The weak topology of any infinite-dimensional normed space is non-metrizable non-bornological but admits a superresolution. Since Fréchet-Urysohn lcs and spaces $C_p(X)$ are b-Baire-like, Proposition 2 applies to get the following

Theorem 1 *A Fréchet-Urysohn lcs is metrizable if and only if it admits a superresolution. $C_p(X)$ is metrizable if and only if $C_p(X)$ admits a superresolution.*

As we have already mentioned a Fréchet-Urysohn lcs in class $\mathfrak{G}$ is metrizable. But only metrizable spaces $C_p(X)$ belong to class $\mathfrak{G}$, see [3]. On the other hand, (this fact might be already known) *under* (MA)+¬(CH) *there exist non-metrizable analytic Fréchet-Urysohn spaces $C_p(X)$*. Indeed, by [1, Theorem II.3.2] the space $C_p(X)$ is Fréchet-Urysohn if and only if X has the *γ-property*, i.e. if for any open cover $\mathcal{R}$ of X such that any finite subset of X is contained in a member of $\mathcal{R}$, there exists an infinite subfamily $\mathcal{R}'$ of $\mathcal{R}$ such that any element of X lies in all but finitely many members of $\mathcal{R}'$. Gerlits and Nagy [7] showed that under (MA) every subset of reals of cardinality smaller than the continuum has the γ-property. Hence under MA+¬(CH) there are uncountable γ-subsets Y of reals, see also [8]. Thus for such Y the space $C_p(Y)$ is non-metrizable separable and Fréchet-Urysohn.

A set of reals A is said to have *strong measure zero* if for any sequence $(t_n)_n$ of positive reals there exists a sequence of intervals $(I_n)_n$ covering A with $|I_n| < t_n$ for each $n \in \mathbb{N}$. The Borel Conjecture states that every strong measure zero set is countable; Laver [11] proved that it is relatively consistent with ZFC that the Borel conjecture is true. *Assuming the Borel Conjecture, every Fréchet-Urysohn separable space $C_p(X)$ is metrizable*. Indeed, since $C_p(X)$ is separable and Fréchet-Urysohn, X admits a weaker separable metric topology ξ and X has the γ-property. Set $Y := (X, \xi)$. Since Y has the γ-property, then $C_p(Y)$ is Fréchet-Urysohn [7]. By [12, Proposition 4 and Theorem 1] the space Y is zero-dimensional. Since Y is metrizable and separable, it is homeomorphic to a subset Z with the γ-property of the Cantor set. On the other hand, Gerlits-Nagy [7] proved that γ-sets have strong measure zero. By the Borel Conjecture any γ-set in the reals is countable. Hence Z is countable. Thus X is countable and $C_p(X)$ is metrizable.

Since Fréchet-Urysohn lcs are b-Baire-like, then every Fréchet-Urysohn lcs with a fundamental sequence of bounded sets must be a (DF)-space. But Fréhct-Urysohn (DF)-spaces are metrizable, [3]. For topological groups the situation is even more striking. The paper [21, Example 1.2] provides non-metrizable Fréchet-Urysohn σ-compact topological groups but for Fréchet-Urysohn hemicompact groups X the situation is different. Next proposition extends Webb's theorem [22, Theorem 5.7], who proved that only finite-dimensional Montel (DF)-spaces enjoy the Fréchet-Urysohn property.

Proposition 3 *Every Fréchet-Urysohn hemicompact topological group X is a locally compact Polish space.*

PROOF. Let $(K_n)_n$ be an increasing sequence of compact sets covering X such that every compact set in X is contained in some K_m. Observe first that X is locally compact: It is enough to show that there exists $n \in \mathbb{N}$ such that K_n is a neighborhood of the unit of X. Let $\mathfrak{F}$ be a base of neighborhoods of the unit of X and assume that no K_n contains a member of $\mathfrak{F}$. For every $U \in \mathfrak{F}$ and $n \in \mathbb{N}$ choose $x_{U,n} \in U \setminus K_n$, and for each $n \in \mathbb{N}$ let $A_n = \{x_{U,n} : U \in \mathfrak{F}\}$. Since $0 \in \overline{A_n}$ for every $n \in \mathbb{N}$, there exists a sequence $(U_{m,n})_m$ in $\mathfrak{F}$ such that $x_{m,n} \to 0$, $m \to \infty$, where $x_{m,n} := x_{U_{m,n},n}$. By [14, Theorem 4] there exists a sequence $(n_k)_k$ of distinct numbers in $\mathbb{N}$ and a sequence $(m_k)_k$ in $\mathbb{N}$ such that $x_{m_k,n_k} \to 0$. Since $\{x_{m_k,n_k} : k \in \mathbb{N}\}$ is contained in some K_p but $x_{m_k,n_k} \notin K_{n_k}$, $k \in \mathbb{N}$, we get a contradiction. Hence X is a locally compact Fréchet-Urysohn group. Since every locally compact Fréchet-Urysohn topological group is metrizable, see e.g. [10, Theorem 2], we conclude that X is analytic. But any analytic Baire topological group is a Polish space [4, Theorem 5.4], and we reach the conclusion. ■

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