Some recent results for controllability of fluid flows

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Abstract

In the late 1980's, J.-L. Lions asked the question about "controllability" of fluid flows. The term " controllability" was not very precise at that time. Of course, there was no hope for obtaining " exact controllability" for Navier-Stokes equations because of dissipativity and irreversability. Concerning Euler equations, he did not know exactly what to expect. He showed that the linearized problem around zero was not controllable, but he knew that this fact did not mean very much. For Navier-Stokes equations he was clearly expecting " approximate controllability". This is still an open problem (for classical boundary conditions) but at the same time, the question is not really relevant : let us suppose that by controlling the system we can reach any neighborhood of a given target at time T. Then what do we do after time T? Anyway Lions clearly thought that this could be an important step in the understanding of how we can drive the Navier-Stokes system.

In the early 1990's, A. Fursikov and O. Imanuvilov showed that the (dissipative) Burger's equation was not approximately controllable. But this did not change Lions' mind because he said that Burger's equation was not controllable because it was too much stable, whereas Navier-Stokes equations are much less stable. His philosophy was that the more a system is unstable, the easier it is controllable. The challenge was exciting and important and a few years after came some very interesting positive answers.We will describe some of these results in this conference.

In 1996, J.-M. Coron proved an exact controllability result for Euler's equations in $2-d$. He then extended this result for proving an approximate controllability result for Navier-Stokes equations (again in $2-d$) when the boundary conditions are not the standard ones but what Coron calls the Navier boundary conditions.

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About the same time, L. Robbiano and G. Lebeau, A. Fursikov and O. Imanuvilov, by different methods, proved a result of "null controllability" for the heat equation. Fursikov-Imanuvilov extended their method to proving for the (controlled) Navier-Stokes system with boundary conditions on the curl of the solution, that you can reach in finite time any stationnary solution (for example), even an unstable one, if the initial condition is not too far from this stationnary solution. This was a real breakthrough and gave rise to a number of extensions later on. The idea of exact controllability to trajectories was there, saying (roughly speaking) that, even if you cannot reach any point in the state space, you can reach any point on the trajectories of the same operator. In a very important article, O. Imanuvilov proved that the same result holds for classical Dirichlet boundary conditions, under rather strong regularity assumptions. A lot of work has been done to improve and extend his result since then and recently, in a work in collaboration by O. Imanuvilov and J.-P. Puel and E. Fernandez-Cara, S. Guerrero, O. Imanuvilov and J.-P. Puel, we have given a rather strong extension of Imanuvilov's first result with systematic proofs of each step of the argument, some of the results being now optimal.

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