Theoretical results for the Navier-Stokes equations and some related systems

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Abstract

In this talk, I will try to recall some contributions of J.L. Lions to the analysis of parabolic systems of the Navier-Stokes kind. I will also indicate some further results.

Thus, let us assume that Ω is a regular bounded domain and let us first consider the following initial/boundary value problem for the Navier-Stokes equations:

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f, \quad \nabla \cdot u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

Here, u and p are the unknowns (the velocity field and the pressure distribution of the fluid), u_t is the time derivative of u, ν is a positive constant, $f = f(x,t)$ and $u^0 = u^0(x)$ are prescribed fields and $(u \cdot \nabla)u$ stands for the vector whose i-th component is

$$\sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} \, .$$

For *reasonable* L^2 data f and u^0 and any $T > 0$, the nonlinear system (1) possesses at least one solution (u,p) with *reasonable* properties. Furthermore, this solution is unique when $N = 2$. Whether or not it is also unique in the three-dimensional case is a major open problem in the theory of nonlinear partial differential equations.

One of the first contributions of J.L. Lions in this field was the following partial uniqueness result: if $N = 3$ and (u,p) is a solution satisfying $u \in L^r(0,T;L^s(\Omega))$ with

$$\frac{2}{r} + \frac{3}{s} \le 1, \quad s > 3, \tag{2}$$

then (u,p) is unique in this class (see [7]).

The limit case $s = 3$ was considered by P.L. Lions and N. Masmoudi in [9] thirty years later. They found that the same conclusion holds for any solution in the space $C^0([0,T];L^3(\Omega))$ (see also [5]).

For nonhomogeneous (i.e. variable density), viscous newtonian and incompressible fluid, the governing equations are the following:

$$\begin{cases} \rho\,(u_t + (u\cdot\nabla)u) - \mu\Delta u + \nabla p = \rho f, \quad \nabla\cdot u = 0 & \text{in } \Omega\times(0,T), \\ \rho_t + u\cdot\nabla\rho = 0 & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u^0(x), \quad \rho(x,0) = \rho^0(x) & \text{in } \Omega. \end{cases} \tag{3}$$

In this system, the unknowns are ρ (the mass density), u and p. Again, μ is a positive constant and ρ^0 is given.

The existence properties of (3) have been analyzed by several authors. Among others, J.L. Lions proved in [8] that, for L^2 data f and u^0 and measurable data ρ^0 satisfying

$$0 < \underline{\rho} \le \rho^0 \le \overline{\rho} < +\infty, \tag{4}$$

the system (3) possesses at least one solution (u,p,ρ) (see also [6]). This was later generalized by J. Simon [12] to the case in which we only assume that

$$0 \le \rho^0 \le \overline{\rho} < +\infty. \tag{5}$$

Other more recent results for (3) are known at present (see for instance [10] and the references therein). However, the uniqueness of solution in natural energy spaces is unknown even when $N = 2$.

Other systems from fluid mechanics considered by J.L. Lions concern (homogeneous and nonhomogeneous) quasi-newtonian fluids. For instance, for a density varying, visco-plastic or dilatant fluid, the equations are the following:

$$\begin{cases} \rho(u_t + (u\cdot\nabla)u) - \nabla\cdot\tau + \nabla p = \rho f, \quad \nabla\cdot u = 0 & \text{in } \Omega\times(0,T), \\ \rho_t + u\cdot\nabla\rho = 0 & \text{in } \Omega\times(0,T), \\ \tau = (\mu + \alpha|D(u)|^{r-2})D(u), \quad D(u) = \nabla u + {}^t\nabla u, \quad r \ge 1, & \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u^0(x), \quad \rho(x,0) = \rho^0(x) & \text{in } \Omega. \end{cases} \tag{6}$$

When ρ is constant, we find a system similar to (1) that was analyzed in [7] and later in [2] and [1] (among others). In particular, if ρ is constant and $r = 1$, we find a homogeneous *Bingham fluid*. The rigorous formulation and global existence of a solution (u,p) were given in the book [3] in this case.

For other more recent results, see [11] and [4].

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