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Metrizability of Precompact Sets; an Elementary Proof

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Abstract. We provide a very short proof of a Cascales-Orihuela's theorem stating that every precompact set in a locally convex space in class $\mathfrak{G}$ (in sense of Cascales-Orihuela) is metrizable.

Metrizabilidad de conjuntos precompactos; una demostración elemental

Resumen. Proporcionamos una demostración muy corta de un teorema de Cascales-Orihuela que establece que todo conjunto precompacto de un espacio localmente convexo de la clase $\mathfrak{G}$ (en el sentido de Cascales-Orihuela) es metrizable.

1 Introduction and preliminaries

In [6, p. 36] Floret (being motivated by earlier results of Grothendieck, Fremlin, De Wilde and Pryce) presented a general version of the Eberlian-Šmulian theorem with many applications. But his result did not include some important classes of locally convex spaces (lcs), and said nothing about metrizability of compact subsets. In [1] Cascales and Orihuela (answering a question of Floret [5]) showed that the weight of any precompact set in an (LM)-space is countable, i.e. precompact sets are metrizable in inductive limits of increasing sequences of metrizable lcs. Pfister and Valdivia, respectively, had shown earlier the same result in (DF)-spaces and dual metric spaces, [9, 11]. In [8] Kąkol and Saxon presented alternative proofs for (LM)-spaces and dual metric spaces. This line of research was continued in [2], where Cascales and Orihuela introduced a large class $\mathfrak{G}$ of lcs including (LF)-spaces and (DF)-spaces and proved (among other things) the following result.

Theorem 1 *Every precompact set in a locally convex space E in class $\mathfrak{G}$ is metrizable.*

Their argument was based on a theorem proved for uniform spaces which involved K-analytic structures connected with ordered families of compact sets. In [10] N. Robertson used the concept of trans-separability to obtain another version of Cascales-Orihuela's result (the concept of trans-separability had already been used in [7] and [9] while studying also metrizability of precompact sets in certain lcs). He proved (using a cardinality argument) that:

(R) *If a lcs E is covered by a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of precompact sets which is ordered, i.e. $A_\alpha \subset A_\beta$ for $\alpha \leq \beta$, then it is trans-separable.*

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Result (R) applies in [10] to show that if the topological dual E' of E endowed with the topology τ_p of the uniform convergence on precompact sets of E satisfies the assumption stated in (R), every precompact set in E is metrizable.

Following Robertson's argument and using Zorn's lemma we provide also another elementary and very short proof of the Theorem above.

Following Cascales and Orihuela [2] we shall say that a lcs E belongs to class $\mathfrak{G}$ if there exists a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of its topological dual E' (called its $\mathfrak{G}$-representation) such that:

(a) $E' = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$

(b) $A_\alpha \subset A_\beta$ when $\alpha \leqslant \beta$

(c) in each A_α, sequences are equicontinuous

where the set $\mathbb{N}$ is endowed with the discrete topology and $\mathbb{N}^{\mathbb{N}}$ with its product topology.

Condition (c) implies that every set A_α is bounded in the strong topology $\beta(E', E)$ of E' and also $\sigma(E', E)$-relatively countably compact.

The class $\mathfrak{G}$ is stable by taking subspaces, separated quotients, completions, countable direct sums and countable products, and contains important classes of spaces like (LM)-spaces (hence (LF)-spaces), dual metric spaces (hence (DF)-spaces), the space of distributions $D'(\Omega)$ and real analytic functions $A(\Omega)$ for open $\Omega \subset \mathbb{R}^{\mathbb{N}}$, etc., see [3, 4].

2 Proof of Theorem

Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E.

For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ put

$$C_{n_1,n_2,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k) \in \mathbb{N}^{\mathbb{N}}, n_j = m_j, j = 1, 2, \ldots, k\}.$$

By $D_{n_1,n_2,\ldots,n_k}$ we denote the polars of $C_{n_1,n_2,\ldots,n_k}$, $k \in \mathbb{N}$. Let P be a precompact set in E. Since the completion of a lcs in class $\mathfrak{G}$ belongs to class $\mathfrak{G}$, we may assume that P is compact.

Claim 1 *For each $\epsilon > 0$ there is a countable subset H_ϵ in E' such that $E' = H_\epsilon + \epsilon(P)^\circ$.*

Otherwise (by Zorn's lemma) there exist an uncountable subset F in E' and $\epsilon > 0$ such that the condition $f - g \in \epsilon(P)^\circ$ for $f, g \in F$ implies $f = g$. By an obvious induction we select a sequence $(n_k)_k$ in $\mathbb{N}$ and a sequence $(f_k)_k$ in E' of different elements with $f_k \in C_{n_1,n_2,\ldots,n_k}$ such that $f_n - f_m \in \epsilon(P)^\circ$ implies $m = n$. Indeed, there exists $n_1 \in \mathbb{N}$ such that $F \cap C_{n_1}$ is uncountable. Choose $f_1 \in F \cap C_{n_1}$. Since $C_{n_1} = \bigcup\{C_{n_1,m_2} : m_2 \in \mathbb{N}\}$, there exists $n_2 \in \mathbb{N}$ such that $(F \setminus \{f_1\}) \cap C_{n_1,n_2}$ is uncountable. Select $f_2 \in (F \setminus \{f_1\}) \cap C_{n_1,n_2}$. Continuing on this manner we obtain inductively the both sequences as desired. Since $f_k \in C_{n_1,n_2,\ldots,n_k}$ for all $k \in \mathbb{N}$, the sequence $(f_k)_k$ is equicontinuous. Indeed, for every $k \in \mathbb{N}$ there exists $\beta_k = (m_n^k)_n \in \mathbb{N}^{\mathbb{N}}$ such that $f_k \in A_{\beta_k}$, where $n_j = m_j^k$ for $j = 1, 2, \ldots, k$. Define $a_n = \max\left\{m_n^k : k \in \mathbb{N}\right\}$ and $\gamma = (a_n) \in \mathbb{N}^{\mathbb{N}}$. Note that $\gamma \geqslant \beta_k$ for every $k \in \mathbb{N}$. Therefore $A_{\beta_k} \subset A_\gamma$, so $f_k \in A_\gamma$ for all $k \in \mathbb{N}$ (by (b) from the definition of the $\mathfrak{G}$-representation). By (c) the sequence $(f_k)_k$ is equicontinuous. Now Ascoli's theorem for $C_c(P)$ applies to select two different natural numbers j, k such that $f_j - f_k \in \epsilon(P)^\circ$, which yields a contradiction. This proves the claim.

Then, since $H := \{H_{n^{-1}} : n \in \mathbb{N}\}$ is countable, the topology τ_H on E of the pointwise convergence on H restricted to P is Hausdorff and metrizable and coincides with the original topology of P. Hence P is metrizable. ■

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