The summability of solutions to variational problems since Guido Stampacchia

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Abstract. Inequalities concerning the integral of $|\nabla u|^2$ on the subsets where $|u(x)|$ is greater than $k$ can be used in order to prove regularity properties of the function $u$. This method was introduced by Ennio De Giorgi e Guido Stampacchia for the study of the regularity of the solutions of Dirichlet problems.

Integrabilidad de soluciones de problemas variacionales desde Guido Stampacchia

Resumen. Adecuadas desigualdades sobre la integral de $|\nabla u|^2$ extendida a los subconjuntos donde $|u(x)|$ es mayor que $k$ pueden ser usadas para obtener propiedades de regularidad de la función $u$. Este método fue introducido por Ennio De Giorgi y Guido Stampacchia para el estudio de la regularidad de las soluciones de problemas de Dirichlet.

1. The Stampacchia method

I recall the following regularity results by Guido Stampacchia, concerning solutions of linear Dirichlet problems.

Let $\Omega$ be a bounded subset of $\mathbb{R}^N$ ($N > 1$) and $L(u) = -\text{div}(M(x)\nabla u)$ be a differential operator, where $M$ is a bounded elliptic matrix. Consider the Dirichlet problem

$$u \in W^{1,2}_0(\Omega) : L(u) = f(x) \in L^\infty(\Omega)$$

(1)

The use of

$$G_k(u) = \begin{cases} 
  u(x) + k, & \text{if } x : u(x) < -k; \\
  0, & \text{if } x : |u(x)| \leq k; \\
  u(x) - k, & \text{if } x : u(x) > k;
\end{cases}$$

as test function implies

$$\alpha \int_\Omega |\nabla G_k(u)|^2 \leq \int_\Omega f G_k(u)$$

(2)
If 
\[ f \in M^m(\Omega), \ m > \frac{2N}{N+2} \] (3)

by Sobolev inequality we have
\[ \left( \int_{\Omega} |G_k(u)|^{2^*} \right)^{\frac{1}{2^*}} \leq c_f |A_k|^{\frac{N+2}{2N} - \frac{1}{m}} \]

where \(|E|\) denotes the measure of the subset \(E\) and
\[ A_k = \{ x \in \Omega : |u(x)| > k \} \]

Then, if \( h > k > 0 \), we have
\[ (h - k)|A_k|^{\frac{1}{2^*}} \leq c_f |A_k|^{\frac{N+2}{2N} - \frac{1}{m}} \]
\[ |A_h| \leq c_f |A_k|^{2^* \left( \frac{N+2}{2N} - \frac{1}{m} \right)} \]
(4)

Here we use the following lemma in order to prove that, in (4),
• if \( 2^* \left( \frac{N+2}{2N} - \frac{1}{m} \right) > 1 \) (that is \( m > \frac{N}{2} \)), there exists \( M > 0 \) such that \(|A_M| = 0\): \( u \) is bounded (\(|u| \leq M\));
• if \( 2^* \left( \frac{N+2}{2N} - \frac{1}{m} \right) < 1 \) (that is \( \frac{2N}{N+2} \leq m < \frac{N}{2} \)), then there exists \( c_0 > 0 \) such that
\[ |A_k| \leq c_0 \frac{c_f k^{m^*}}{k^{m^*}} \] \(^\blacksquare\)

\( u \) belongs to the Marcinkiewicz space \( M^{m^*}(\Omega) \).

**Lemma 1 (Stampacchia’s Lemma)** Let \( \phi(t) \) be a positive, decreasing real function such that
\[ h > k \Rightarrow \phi(h) \leq C \frac{\phi(k)^\theta}{(h-k)^a} \quad 0 < \theta < 1, \ a > 0 \] (5)
then there exist \( c_0 \) and \( k_0 \) such that
\[ \phi(k) \leq c_0 \frac{C \phi(k)^\theta}{k^{a\theta}}, \ k \geq k_0 \]
if
\[ h > k \Rightarrow \phi(h) \leq C \frac{\phi(k)^\lambda}{(h-k)^a}, \ \lambda > 1 \]
(6)
then there exist \( M \) such that
\[ \phi(M) = 0. \] \(^\blacksquare\)

We repeat the results proved in the previous page:

**Theorem 1 (Stampacchia’s regularity)** The solution \( u \) of the Dirichlet problem (1) is bounded, if \( f \in M^m(\Omega), \) with \( m > \frac{N}{2} \) and \( u \) belongs to \( M^{m^*}(\Omega) \), if \( f \in M^m(\Omega), \) with \( \frac{2N}{N+2} \leq m < \frac{N}{2} \). \(^\blacksquare\)
Thanks to interpolation, the following theorem follows from the linearity of the differential operator.

**Theorem 2 (Stampacchia’s summability)** If \( f \in M^m(\Omega) \), with \( \frac{2N}{N+2} \leq m < \frac{N}{2} \), then \( u \) belongs to \( L^{m^{**}}(\Omega) \). ■

Developments of this method can be found in in [6], [8], [10], [17], [21], [22], [25], [26], [27], [18], [24].

### 1.1. Nonlinear operators

Consider, now, the nonlinear differential operator \( A(v) = -\text{div}(a(x, v, \nabla v)) \) in \( W^{1,p}_0(\Omega) \) \((p > 1)\), with the usual Leray-Lions assumptions (see [23]), and the Dirichlet problem

\[
u \in W^{1,p}_0(\Omega) : A(u) = f(x)
\]

For sake of simplicity, we still take \( p = 2 \):

\[
u \in W^{1,2}_0(\Omega) : A(u) = f(x) \in L^{\frac{2N}{N+2}}(\Omega)
\] (7)

The proofs of Theorem 1 still hold.

**Theorem 3 (Stampacchia’s regularity)** The solution \( u \) of the Dirichlet problem (7) is bounded, if \( f \in M^m(\Omega) \), with \( m > \frac{N}{2} \) and \( u \) belongs to \( M^{m^{**}}(\Omega) \), if \( f \in M^m(\Omega) \), with \( \frac{2N}{N+2} \leq m < \frac{N}{2} \). ■

Theorem 2 still holds, with a different proof (powers of as \( u \) test functions): see [14], [15] for the proof and applications (developments in [9]).

**Theorem 4 (Summability)** If \( f \in L^m(\Omega) \), with \( m > \frac{N}{2} \), then \( u \) is bounded; if \( f \in M^m(\Omega) \), with \( \frac{2N}{N+2} \leq m < \frac{N}{2} \), then \( u \) belongs to \( L^{m^{**}}(\Omega) \).

**Proof.** Use

\[
v = \frac{|T_k(u)|^{2\lambda}T_k(u)}{2\lambda + 1}, \quad \lambda = \frac{-mN + 2N - 2m}{4m - 2N}; \quad k > 0
\]

as test function in the weak formulation of (7) and Sobolev inequality:

\[
\frac{1}{2\lambda + 1} \int_\Omega a(x, u, \nabla u)\nabla(|T_k(u)|^{2\lambda}T_k(u)) \geq c_1(\lambda)\alpha \left( \int_\Omega |T_k(u)|^{(\lambda+1)2^*} \right)^{\frac{1}{2^*}}.
\]

Then the Hölder inequality implies that

\[
\left( \int_\Omega |T_k(u)|^{(\lambda+1)2^*} \right)^{\frac{1}{2^*}} \leq c_2(\alpha, \lambda) \left( \int_\Omega |T_k(u)|^{(2\lambda+1)m'} \right)^{\frac{1}{m'}} \|f\|_{L^m(\Omega)}.
\]

The definition of \( \lambda \) gets \((\lambda + 1)2^* = m'(2\lambda + 1)\), and

\[
\|T_k(u)\|_{L^{m^{**}}(\Omega)} \leq c_3(\alpha, m) \|f\|_{L^m(\Omega)}.
\]

Now, if \( k \to \infty \), the Fatou Lemma implies

\[
\|u\|_{L^{m^{**}}(\Omega)} \leq c_3(\alpha, m) \|f\|_{L^m(\Omega)}. \tag{8}
\]
1.2. Minima of functionals

Consider, now, the following functional of the Calculus of Variations

$$J(v) = \int_\Omega j(x, v, \nabla v) - \int_\Omega f v$$

in $W^{1,p}_0(\Omega)$ ($p > 1$), with the usual assumptions on $j$ (see [20], [19]), and the minimization problem

$$u \in W^{1,p}_0(\Omega) : J(u) \leq J(v), \; \forall v \in W^{1,p}_0(\Omega)$$

For sake of simplicity, we still take $p = 2$:

$$u \in W^{1,2}_0(\Omega) : J(u) \leq J(v), \; \forall v \in W^{1,2}_0(\Omega) \quad (9)$$

Let

$$T_k(u) = \begin{cases} 0, & \text{if } x : u(x) < -k; \\ u, & \text{if } x : |u(x)| \leq k; \\ +k, & \text{if } x : u(x) > k; \end{cases}$$

If we take $v = u - G_k(u) = T_k(u)$, we get again the inequality (2), so that the proof of Theorem 1 still holds.

**Theorem 5 (Stampacchia’s regularity)** The minima $u$ of the minimization problem (9) is bounded, if $f \in M^m(\Omega)$, with $m > \frac{N}{2}$ and $u$ belongs to $M^{m^{*}r}(\Omega)$, if $f \in M^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$.

If we assume that $f \in M^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$ the choice

$$v = u - \frac{[T_k(u)]^{2\lambda}T_k(u)}{2\lambda + 1}, \quad \lambda = \frac{-mN + 2N - 2m}{4m - 2N}$$

it is not useful.

**Theorem 6 (Summability [16])** The minimum $u$ of the minimization problem (9) belongs to $L^{m^{*}r}(\Omega)$, if $f \in L^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$.

**Proof.** We take again $v = u - G_k(u) = T_k(u)$ and we get the inequality (2). Then

$$\int_{\{x \in \Omega : |u(x)| \geq k\}} |\nabla u|^2 \leq \left( \int_{\{x \in \Omega : |u(x)| \geq k\}} |f|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{N}} .$$

The previous inequality implies that, for every $k > 0$, and $\lambda = \frac{-mN + 2N - 2m}{4m - 2N}$ as in Theorem 4

$$k^{2\lambda-1} \int_{\{x \in \Omega : |u(x)| \geq k\}} |\nabla u|^2 \leq k^{2\lambda-1} \left( \int_{\{x \in \Omega : |u(x)| \geq k\}} |f|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{N}} .$$

Then, as starting point of the proof, we write the previous inequality as

$$\sum_{j=k}^{\infty} \int_{\{x \in \Omega : |u(x)| \geq j+1\}} |\nabla u|^2 \leq k^{2\lambda-1} \left( \int_{\{x \in \Omega : |u(x)| \geq k\}} |f|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{N}} .$$
which gets (formally)

\[ \sum_{k=0}^{\infty} k^{2\lambda-1} \sum_{j=k(x \in \Omega : |u(x)| < j+1)} \int_{|u| \geq k} |\nabla u|^2 \leq \sum_{k=0}^{\infty} k^{2\lambda-1} \left( \int_{|u| \geq k} \left| f \right|^\frac{2N}{N+2} \right)^\frac{N+2}{N}. \]

But it is possible (but not easy) to prove that

\[ \sum_{k=0}^{\infty} k^{2\lambda-1} \sum_{j=k(x \in \Omega : |u(x)| < j+1)} \int_{|u| \geq k} |\nabla u|^2 \approx \int_{\Omega} |u(\lambda+1)^m| \cdot \| f \|_{L^m(\Omega)} \]

and

\[ \sum_{k=0}^{\infty} k^{2\lambda-1} \left( \int_{|u| \geq k} \left| f \right|^\frac{2N}{N+2} \right)^\frac{N+2}{N} \approx \left( \int_{\Omega} |u(\lambda+1)^m| \right)^\frac{1}{m} \]

in order to get again the inequality (8) and show the summability of the minimum \( u \). ■

Developments of above method method ([16]) can be found in [10] (regularity of minimizing sequences) and in [7] (parabolic equations).

## 2. Singular data ([4])

### 2.1. Dirichlet problems in large Sobolev spaces

In this subsection, I report some results concerning Marcikiewicz estimates on the solutions of Dirichlet problems with irregular data. Aim of Theorem 7 is to give an easier and shorter proof of some results of [1].

Consider again the nonlinear differential operator \( A(v) = -\text{div}(a(x, v, \nabla v)) \) and the boundary value problem

\[
\begin{cases}
-\text{div}(a(x, u, \nabla u)) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\tag{10}
\]

where on the right hand side we assume only that \( f \in L^1(\Omega) \).

The existence and properties of solutions is proved in [11], [12], [2], [13], [3]. Moreover in [12] is proved that \( u \) belongs to \( W^{1,m^*}(\Omega) \), if \( f \) belongs to \( L^m(\Omega) \), if \( 1 < m < \frac{2N}{N+2} \).

Now we shall discuss the regularity of \( u \) if

\[ f \in M^m(\Omega), \quad 1 < m < \frac{2N}{N+2} \tag{11} \]

**Theorem 7** If \( f \) belongs to \( M^m(\Omega), \ 1 < m < \frac{2N}{N+2} \), the weak solutions \( u \) of (10) belong to \( M^{m^*}(\Omega) \) and \( \nabla u \in M^{m^*}(\Omega) \).

**Proof.** We cannot use the approach of Stampacchia, since it is not possible to use \( u \) (and \( G_k(u) \)) as test function in the Dirichlet problem, because \( |\nabla u|^2 \) does not belong to \( L^1(\Omega) \).

Use (formally) as test function \( T_{h-k}[G_k(u)] \). Thus

\[
\alpha \int_{\Omega} |\nabla T_{h-k}[G_k(u)]|^2 \leq \int_{\Omega} fT_{h-k}[G_k(u)] \tag{12}
\]

\[ \alpha S^2 (h - k)^2 |A_k|^2 \leq c_f (h - k)|A_k|^{1 - \frac{m}{m}} \]
Here we use the Lemma 1 with $\theta = \left(1 - \frac{1}{m}\right) \frac{2^{*}}{2}$, so that $\frac{2^{*}}{2(1 - \theta)} = m^{**}$: $u$ belongs to $M^{m^{**}}(\Omega)$.

Moreover, if in (9) we take $h = k + 1$ we have

$$
\alpha \int_{\Omega} |\nabla u|^2 \leq \int_{A_k} |f| \leq c_f |A_k|^{1 - \frac{2}{m^{**}}} \leq C_0 \frac{c_f}{k^{m^{**}}(1 - \frac{2}{m^{**}})}
$$

where

$$
B_k = \{x \in \Omega : k \leq |u(x)| < k + 1\}
$$

and $A_0 = \Omega$. Thus

$$
\alpha \int_{\Omega} |\nabla T_k(u)|^2 \leq |\Omega| + \sum_{i=1}^{i=k-1} C_0 \frac{c_f}{i^{m^{**}}(1 - \frac{2}{m^{**}})}
$$

Remark that

$$
0 \leq m^{**}(1 - \frac{1}{m}) < 1 \iff 1 \leq m < \frac{2N}{N + 2}
$$

and $(0 < \theta < 1)$

$$
\frac{(k - 1)^{1-\theta}}{1 - \theta} > 1 + \frac{(k - 1)^{1-\theta} - 1}{1 - \theta} = 1 + \int_{1}^{k-1} \frac{1}{t^\theta} = 1 + \sum_{j=1}^{k-2} \int_{j}^{j+1} \frac{1}{t^\theta} > 1 + \sum_{j=1}^{k-2} \frac{1}{(j + 1)^\theta} = \sum_{i=1}^{i=k-1} \frac{1}{i^\theta}
$$

So, for $k \geq 1$,

$$
\alpha \int_{\Omega} |\nabla T_k(u)|^2 \leq |\Omega| + c_3 e_f (k - 1)^{1 - m^{**}(1 - \frac{2}{m^{**}})} \leq C_f k^{1 - m^{**}(1 - \frac{2}{m^{**}})}
$$

(13)

Here we follow a technique of [2]. Estimate (15) implies also

$$
t^2 \operatorname{meas}(A_k \cap \{|\nabla u| > t\}) \leq \int_{A_k} |\nabla u|^2 \leq c_2 e_f k^{1 - m^{**}(1 - \frac{2}{m^{**}})}
$$

On the other hand

$$
\operatorname{meas}\{|\nabla u| > t\} \leq \operatorname{meas}\{|\nabla u| > t, |u| \leq k\} + \operatorname{meas}\{|u| > k\}
$$

$$
\leq c_1 \frac{k^{1 - m^{**}(1 - \frac{2}{m^{**}})}}{t^2} + c_2 \frac{1}{k^{m^{**}}}
$$

Note that

$$
m^{**} \left(1 - \frac{1}{m}\right) = \frac{(m - 1)N}{N - 2m}, \quad 1 - m^{**} \left(1 - \frac{1}{m}\right) = \frac{2N - m(N + 2)}{N - 2m} \in (0, 1]
$$

The minimization with respect to $k$ gives ($k = t \frac{N - 2m}{N - 2m}$)

$$
\operatorname{meas}\{|\nabla u| > t\} \leq \frac{\tilde{C}_f}{t^{m^{**}}}
$$

as desired. ■
Consider the set \( T^{1,2}_0(\Omega) \) of all functions \( u \) which are almost everywhere finite and such that \( T_k(u) \in W^{1,2}_0(\Omega) \) for every \( k > 0 \). For every \( u \in T^{1,2}_0(\Omega) \) there exists a measurable function \( \Phi : \Omega \to \mathbb{R}^N \) such that \( \nabla T_k(u) = \Phi \chi_{\{|u| \leq k\}} \) a.e. in \( \Omega \). This function \( \Phi \), which is unique up to almost everywhere equivalence, will be denoted by \( \nabla u \). Note that \( \nabla u \) coincides with the distributional gradient of \( u \) whenever \( u \in T^{1,2}_0(\Omega) \cap L^1_{loc}(\Omega) \) and \( \nabla u \in L^1_{loc}(\Omega, \mathbb{R}^N) \).

**Definition 1** Let \( f \in L^1(\Omega) \). A function \( u \in T^{1,2}_0(\Omega) \) is a \( T \)-minimum for the functional

\[
J(v) = \int_\Omega j(x, \nabla v) - \int_\Omega f v
\]

if, for every \( \varphi \) in \( W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \) and every \( k > 0 \),

\[
\int_\Omega j(x, \nabla \varphi + \nabla T_k[u - \varphi]) \leq \int_\Omega j(x, \nabla \varphi) + \int_\Omega f T_k[u - \varphi], \tag{14}\]

**Theorem 8 ([5])** There exists a \( T \)-minimum \( u \) of \( J(v) \) such that

\[
\int_\Omega |\nabla T_k(u)|^2 \leq k \left( \frac{||f||_{L^1(\Omega)}}{\alpha} \right) \quad (k > 0), \tag{15}\]

\[
\int_{B_{h,k}} |\nabla u|^2 \leq \frac{1}{\alpha} \int_{A_h} |f| \quad (h, k > 0),
\]

where

\[
B_{h,k} = \{ x \in \Omega : h \leq |u(x)| < h + k \},
\]

\[
A_h = \{ x \in \Omega : h \leq |u(x)| \}.
\]

and \( u \in W^{1,q}_0(\Omega) \), \( q < \frac{N}{N-2} \).

Moreover (Marcikiewicz framework) \( u \) belongs to \( M^{\frac{N}{N-2}}(\Omega) \) and \( \nabla \) belongs to \( M^{\frac{N}{N-2}'}(\Omega) \). ■

If, in (14), we write \( h - k \) instead of \( k \) and take \( \varphi = T_k(u) \), then

\[
\int_\Omega j(x, \nabla T_k(u) + \nabla T_{h-k}[u - T_k(u)]) \leq \int_\Omega j(x, \nabla T_k(u)) + \int_\Omega f T_{h-k}[u - T_k(u)],
\]

which implies the inequality (12), so that we have we can prove the following results.

**Theorem 9** If \( f \) belongs to \( M^m(\Omega) \), \( 1 < m < \frac{2N}{N+2} \), then the \( T \)-minimum \( u \) belongs to \( M^{m^*}(\Omega) \) and \( \nabla u \in M^{m^*}(\Omega) \). ■

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References


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