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# Unitary sequences and classes of barrelledness

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**Abstract.** It is well known that some dense subspaces of a barrelled space could be not barrelled. Here we prove that dense subspaces of $l_\infty(\Omega, X)$ are barrelled (unordered Baire-like or $p-$barrelled) spaces if they have "enough" subspaces with the considered barrelledness property and if the normed space $X$ has this barrelledness property.

These dense subspaces are used in measure theory and its barrelledness is related with some sequences of unitary vectors.

### Sucesiones unitarias y clases de tonelación

**Resumen.** Es bien conocido que existen espacios tonelados con subespacios densos no tonelados. Aquí se prueba que los subespacios densos de $l_\infty(\Omega, X)$ son espacios tonelados (unordered Baire-like o $p-$tonelados) si tienen "suficientes" subespacios con la propiedad de tonelación considerada y si el espacio normado $X$ tiene esa propiedad de tonelación.

Estos subespacios densos se utilizan en teoría de la medida y su clase de tonelación está relacionada con ciertas sucesiones de vectores unitarios.

## 1. Preliminaries

Along this paper $\Omega$ will denote a non void set, $X$ a normed space over the field $\mathbb{K}$ of real or complex numbers, $l_\infty(\Omega, X)$ the linear space over $\mathbb{K}$ of all those functions $f : \Omega \longrightarrow X$ such that the set $\{\|f(\omega)\| : \omega \in \Omega\}$ is bounded, equipped with the supremum norm $\|f\|_\infty = \sup\{\|f(\omega)\| : \omega \in \Omega\}$, $bcs(\Omega, X)$ the linear subspace of $l_\infty(\Omega, X)$ of all those functions $f \in l_\infty(\Omega, X)$ countably supported and $c_0(\Omega, X)$ the linear subspace of $bcs(\Omega, X)$ of all those functions $f : \Omega \longrightarrow X$ such that for each $\varepsilon > 0$ the set $\{\omega \in \Omega : \|f(\omega)\| > \varepsilon\}$ is finite or empty.

Let us recall that a (Hausdorff) locally convex space $E$ is barrelled if each barrel (i.e. each absorbing, closed and absolutely convex set) in $E$ is a neighborhood of the origin (see [14, Definition 4.1.1]).

A $p-$net in a vector space $Y$ (see [1]) is a family $\mathcal{W} = \{E_t : t \in T_p\}$ of linear subspaces of $Y$, with $\mathbb{T}_p = \bigcup_{k=1}^{p} \mathbb{N}^k$, such that $Y = \bigcup_{n\in\mathbb{N}} E_n$, $E_n \subset E_{n+1}$, $E_t = \bigcup_{n\in\mathbb{N}} E_{t,n}$ and $E_{t,n} \subset E_{t,n+1}$, for $t \in \mathbb{T}_{p-1}$ and $n \in \mathbb{N}$.

A (Hausdorff) locally convex space $E$ is barrelled of class $p$ ($p-$barrelled for short) if given a $p-$net $\mathcal{W} = \{E_t : t \in \mathbb{T}_p\}$ there is a $t \in \mathbb{N}^p$ such that $E_t$ is barrelled and dense in $E$. The barrelled spaces of class 1 were introduced by Valdivia in [23] with the name suprabarrelled spaces, also called ($db$)-spaces in [15] and [20].

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Other definitions of barrelled spaces related to the Banach-Steinhaus theorem or with the closed graph theorem may be found in [6, Theorems 1.1.4, 1.1.8, 3.2.2 and 3.2.4].

It has been discovered that some of the classical barrelled functional spaces are barrelled spaces of class $p$. For instance, Dieudonné (*cf.* [25, p. 133]) proved that $l_0^\infty$ (*i.e.* the linear subspace of $l_\infty$ formed by the sequences taking finitely many different values) is barrelled. The barrelledness of $l_0^\infty$ was also pointed out independently by Saxon [19]. If $\mathcal{A}$ is a ring of subsets of $\Omega$ and $l_0^\infty(\mathcal{A})$ is the linear hull with coefficients in $\mathbb{K}$ of the characteristic functions $\mathcal{X}_A$, $A \in \mathcal{A}$, endowed with the supremum norm, Schachermayer [21] noticed that $l_0^\infty(\mathcal{A})$ is barrelled if and only if its dual $ba(\mathcal{A})$, the vector space over $\mathbb{K}$ of the bounded finitely additive scalar measures defined on $\mathcal{A}$ equipped with the supremum norm, verifies the Nikodým boundedness theorem ([2, p. 80]).

If $\mathcal{A}$ is a $\sigma$-algebra Valdivia noticed that $l_0^\infty(\mathcal{A})$ is suprabarrelled [23] and Ferrando and López Pellicer found that $l_0^\infty(\mathcal{A})$ is $p-$ barrelled [4]. Some other strong barrelledness properties of $l_0^\infty(\mathcal{A})$ and applications may be found in [5], [8], [9], [11], [16] and [17].

It was proved in [13] that if $\Omega$ is countable infinite then $c_0(\Omega, X)$ is barrelled if and only if $X$ is barrelled. For an infinite set $\Omega$, it has been established in [7] that $c_0(\Omega, X)$ is barrelled, ultrabornological or unordered Baire-like ( [22]) if and only if $X$ is barrelled, ultrabornological or unordered Baire-like, respectively. In [12] it has been proved that $c_0(\Omega, X)$ is $1-$ barrelled if and only if $X$ is $1-$ barrelled.

The aim of this paper is to prove that $c_0(\Omega, X)$, $bcs(\Omega, X)$ and a wide class of subspaces of $bcs(\Omega, X)$ are (barrelled) $p-$barrelled if and only if $X$ is (barrelled) $p-$barrelled.

In what follows $supp(f)$ means the support of $f$, i.e. $supp(f) = \{x \in \Omega : f(x) \neq 0\}$. We are going to use the classical notation given, for instance, in [2] and [25]. The linear hull of a subset $A$ of a linear space $E$ will be denoted by $\langle A \rangle$.

If $E$ is a linear subspace of $bcs(\Omega, X)$ we will denote by $\mathcal{S}_E$ the family of all sequences $\{f_n : n \in \mathbb{N}\}$ such that $f_n \in E$, $\|f_n\|_\infty = 1$ for each $n = 1, 2, \ldots$ and whose support verify one of the following conditions:

a) $supp(f_n) \cap supp(f_m) = \emptyset$ if $n \neq m$

b) there is a countable set $\{w_1, w_2, \ldots, w_n, w_{n+1}, \ldots\} \subset \Omega$ such that $supp(f_n) \subset \{w_{n+1}, w_{n+2}, \ldots\}$, for $n = 1, 2, \ldots$

If $f \in E$ and $\Gamma \subset \Omega$ then $P_\Gamma f$ is the element of $bcs(\Omega, X)$ such that $(P_\Gamma f)(x) = f(x)$ if $x \in \Gamma$ and $(P_\Gamma f)(x) = 0$ when $x \notin \Gamma$. We will define $E(\Gamma) = \{f \in E : supp(f) \subset \Gamma\}$ and, in particular $bcs(\Gamma, X)$ $= \{f \in bcs(\Omega, X) : supp(f) \subset \Gamma\}$.

We will denote by $\mathcal{B}$ the family of linear subspaces of $bcs(\Omega, X)$ such that if $E \in \mathcal{B}$ and $\Delta \subset \Gamma \subset \Omega$, being $\Delta$ finite and $\Gamma$ countable, then $bcs(\Delta, X) \subset E(\Gamma) = P_\Gamma(E)$. Then $E = E(\Gamma) + E(\Omega \diagdown \Gamma)$.

## 2. Barrelledness

In the family $\mathcal{B}$ we are going to consider the family $\mathcal{B}_0$ such that the locally convex vector space $E \in \mathcal{B}$ belongs to $\mathcal{B}_0$ if given a sequence $\{f_n : n \in \mathbb{N}\} \in \mathcal{S}_E$ there exists a barrelled space $(F, \tau)$ such that $F \subset E$, $\{f_n : n \in \mathbb{N}\}$ is bounded in $(F, \tau)$ and $\tau$ is a locally convex topology finer than the topology induced in $F$ by the topology of $E$.

**Lemma 1** *If $E \in \mathcal{B}_0$ and $Q$ is a barrel in $E$ there exists a finite set $\Delta$ (possibly empty) such that $Q$ absorbs the unit ball of $E(\Omega \diagdown \Delta)$.*

PROOF. We assert that there is a countable set $\Delta = \{w_1, w_2, \ldots\}$ such that $Q$ absorbs the closed unit ball of $E(\Omega \diagdown \Delta)$. In fact, if this were not true, there would be a $f_1 \in E$ with $\|f_1\|_\infty = 1$ and $f_1 \notin Q$. By the hypotesis and the countability of $\Delta_1 = supp(f_1)$ we deduce the existence of $f_2 \in E(\Omega \diagdown \Delta_1)$ with $\|f_2\|_\infty = 1$ and $f_2 \notin 2Q$. Once again, as the set $\Delta_2 = supp(f_2)$ is countable there exists $f_3 \in E(\Omega \diagdown \{\Delta_1 \cup \Delta_2\})$ with $\|f_3\|_\infty = 1$ and $f_3 \notin 3Q$.

By induction we would obtain a sequence $\{f_n : n \in \mathbb{N}\} \in \mathcal{S}_E$. Then, by hypotesis, there exists a barrelled space $(F, \tau)$ being $F \subset E$ and $\tau$ a locally convex topology finer than the topology induced in $F$

by the topology of $E$, such that $\{f_n : n \in \mathbb{N}\}$ is bounded in $(F, \tau)$ . Therefore $Q \cap F$ is a $0-$ neighborhood in $(F, \tau)$ and then, by boundedness, there exists a $p$ such that $\{f_n : n \in \mathbb{N}\} \subset pQ$. From this relation follows the contradiction $f_p \in pQ$.

Therefore, there exists a countable set $\Delta = \{w_1, w_2, \ldots, w_n \ldots\}$ such that $Q$ absorbs the closed unit ball of $E(\Omega \diagdown \Delta)$.

Now we are going to prove that there exists a natural number $i$ such that $Q$ absorbs the closed unit ball of $E(\{w_{i+1}, w_{i+2}, \ldots\})$.

If this were not true, there would exist a sequence $\{f_n : n \in \mathbb{N}\}$ with $f_n \in E(\{w_{n+1}, w_{n+2}, \ldots\})$, $\|f_n\|_\infty = 1$ and $f_n \notin nQ$ for each $n = 1, 2, \ldots$ But the sequence $\{f_n : n \in \mathbb{N}\} \in S_E$ and then, as in the preceding case, we would obtain a $q \in \mathbb{N}$ such that $\{f_n : n \in \mathbb{N}\} \subset qQ$. This last inclusion contains the contradiction $f_q \in qQ$, which proves that there exists a natural number $i$ such that $Q$ absorbs the closed unit ball of $E(\{w_{i+1}, w_{i+2}, \ldots\})$.

Finally, we have obtained that if $\Delta = \{w_1, w_2, \ldots w_i\}$ then $Q$ absorbs the closed unit ball of $E(\Omega \diagdown \Delta) = E(\Omega \diagdown \{w_1, w_2, \ldots\}) + E(\{w_{i+1}, w_{i+2}, \ldots\})$ ■

**Proposition 1** *Suppose that $E \in \mathcal{B}_0$. Then $E$ is barrelled if and only if $X$ is barrelled.*

PROOF. If $Q$ is a barrel in $E$ then by Lemma 1, there exists a finite set $\Delta$ such that $Q$ absorbs the unit ball of $E(\Omega \diagdown \Delta)$. The barrel $Q$ also absorbs the unit ball of the barrelled space $E(\Delta) = X^\Delta$ (see [6, Proposition 1.1.13]). From the isomorphism between $E$ and $E(\Omega \diagdown \Delta) \times E(\Delta)$ it follows that $Q$ is a neighborhood of zero in $E$.

Conversely, if $E$ is barrelled and $p \in \Omega$ then from the isometry between $X$ and $E(\{p\}) = E / E(\Omega \diagdown \{p\})$ it follows from [6, Proposition 1.1.9] that $X$ is barrelled. ■

A locally convex space $E$ is unordered Baire-like ([22]) if given in $E$ a countable covering $\{A_n, n \in \mathbb{N}\}$ of closed absolutely convex susbsets of $E$, there exists an $A_n$ which is neighbourhood of zero in $E$.

In the family $\mathcal{B}$ we are going to consider the family $\mathcal{B}_{ub}$ such that the locally convex vector space $E \in \mathcal{B}$ belongs to $\mathcal{B}_{ub}$ if given a sequence $\{f_n : n \in \mathbb{N}\} \in \mathcal{S}_E$ there exists an unordered Baire-like space $(F, \tau)$ such that $F \subset E$, $\{f_n : n \in \mathbb{N}\}$ is bounded in $(F, \tau)$, and $\tau$ is a locally convex topology finer than the topology induced in $F$ by the topology of $E$.

**Lemma 2** *Let $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ be a sequence of absolutely convex and closed subsets of $E$ such that $E = \cup_{n \in \mathbb{N}} \langle V_n \rangle$. Suppose that $E \in \mathcal{B}_{ub}$.*

*Then there exists a subfamily $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ of $\mathcal{V}$ and a sequence $\{\Delta_n : n \in \mathbb{N}\}$ of finite subsets of $\Omega$ such that, for every $n \in \mathbb{N}$, $E(\Omega - \Delta_n) \subset \langle W_n \rangle$ and*

$$E = \cup_{n \in \mathbb{N}} \langle W_n \rangle$$

PROOF. First we are going to prove that there exists $m \in \mathbb{N}$ and a countable subset $\Delta_m$ such that

$$E(\Omega - \Delta_m) \subset \langle V_m \rangle.$$

In fact, if this were not true we would find a sequence $\{f_n : n \in \mathbb{N}\}$ of unitary vectors in $E(\Omega)$ such that

$$f_1 \notin \langle V_1 \rangle$$

and

$$f_n \in E(\Omega - \cup_{i=1}^{n-1} \Delta_i) - \langle V_n \rangle, \qquad n = 2, 3, \ldots$$

where $\Delta_i = \text{supp}(f_i)$ and $\| f_i \|_\infty = 1$ for each $i \in \mathbb{N}$.

Then, by hypothesis, there exists an unordered Baire-like space $(F, \tau)$ such that $F \subset E$, $\{f_n : n \in \mathbb{N}\}$ is bounded in $(F, \tau)$ and $\tau$ is a locally convex topology finer than the topology induced in $F$ by the topology of $E$. Therefore, there exists a $V_m$ that contains a nieghbourhood of zero in $(F, \tau)$, implying that the bounded set $\{f_n : n \in \mathbb{N}\}$ is contained in $\langle V_m \rangle$. The inclusion $\{f_n : n \in \mathbb{N}\} \subset \langle V_m \rangle$ contains the contradiction $f_m \in \langle V_m \rangle$, proving our first observation.

Therefore, from this property and [22, Theorem 4.1] we have that there exists a subfamily $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ of $\mathcal{V}$ and a sequence $\{\Delta_n : n \in \mathbb{N}\}$ of countable subsets of $\Omega$ such that $E(\Omega - \Delta_n) \subset \langle W_n \rangle$, for each $n \in \mathbb{N}$, and $E = \bigcup_{n\in\mathbb{N}} \langle W_n \rangle$.

From this first property it follows that it is enough to prove the lemma for $\Omega = \mathbb{N}$.

In this case we are going to prove that there exists some natural number $m$ such that

$$E(\mathbb{N} - \{1, 2, \dots, m\}) \subset \langle V_m \rangle.$$

If this property were not true we would find a sequence $\{f_n : n \in \mathbb{N}\}$ of unitary vectors in $E(\mathbb{N})$ such that

$$f_n \in E(\mathbb{N} - \{1, 2, \dots, n\}) - \langle V_n \rangle$$

and we would have that the sequence $\{f_n : n \in \mathbb{N}\} \subset \mathcal{S}_E$. By hypothesis, there exists an unordered Baire-like space $(F, \tau)$ such that $F \subset E$, $\{f_n : n \in \mathbb{N}\}$ is bounded in $(F, \tau)$ and $\tau$ is a locally convex topology finer than the topology induced in $F$ by the topology of $E$. Exactly as in the preceding case we would obtain the contradiction $f_p \in \langle V_p \rangle$, proving the second property we are looking for.

These two properties imply that there exists $m \in \mathbb{N}$ and a finite subset $\Delta_m$ such that

$$E(\Omega - \Delta_m) \subset \langle V_m \rangle$$

and, then, from [22, Theorem 4.1] it follows the lemma. ■

**Proposition 2** *Suppose that $E \in \mathcal{B}_{ub}$. Then $E$ is unordered Baire-like if and only if $X$ is unordered Baire-like.*

PROOF. If $E$ is unordered Baire-like and $p \in \Omega$, then from the isometry between $X$ and $E(\{p\}) = E/E(\Omega - \{p\})$ it follows from [6, Proposition 1.3.6] that $X$ is unordered Baire-lke.

Conversely, if $X$ is unordered Baire-like and $E$ were not unordered Baire-like, then there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of absolutely convex and closed subsets of $E$ such that

$$E = \bigcup \{V_n, n \in \mathbb{N}\}$$

and each $V_n$ is not a neighbourhood of zero in the barrelled space $E$ (see Proposition 1). Then, by barrelledness, we have that

$$E \not\subseteq \langle V_n \rangle, \qquad n \in \mathbb{N}.$$

From these relations and Lemma 2 we deduce that there exists a subsequence $\{W_n : n \in \mathbb{N}\}$ of $\{V_n : n \in \mathbb{N}\}$ and a sequence $\{\Delta_n : n \in \mathbb{N}\}$ of finite subsets of $\Omega$ such that

$$\begin{aligned} E &= \bigcup_{n\in\mathbb{N}} \langle W_n \rangle \\ E &\not\subseteq \langle W_n \rangle, \qquad n \in \mathbb{N} \\ E(\Omega - \Delta_n) &\subset \langle W_n \rangle, \qquad n \in \mathbb{N} \end{aligned}$$

It is clear that we have for each $n \in \mathbb{N}$ that

$$E(\Delta_n) \not\subset \langle W_n \rangle$$

and then there exists for each $n$ some $\delta_n \in \Delta_n$ such that

$$E(\{\delta_n\}) \not\subset \langle W_n \rangle.$$

We consider the equivalence relation $R$ in $\mathbb{N}$ defined by the equality $\delta_m = \delta_n$ (i.e. $mRn$ if $\delta_m = \delta_n$). This relation defines a partition $\{F_n, n \in \mathbb{P}\}$ in $\mathbb{N}$, where $\mathbb{P}$ is a finite or countable subset of $\mathbb{N}$.

Let $\{w_n : n \in \mathbb{P}\}$ be the finite or countable subset of $\Omega$ such that $w_n = \delta_s$, being $s$ an arbitrary element of $F_n$. We may rewrite the relations $E(\{\delta_n\}) \not\subset \langle W_n \rangle$, $n \in \mathbb{N}$, in the form:

$$E(\{w_n\}) \not\subset \langle W_n \rangle, \quad m \in F_n, \quad n \in \mathbb{P}.$$

We have that the space $X$ is unordered Baire-like and that $E(\{w_n\})$ and $X$ are isometric. Therefore

$$E(\{w_n\}) \not\subset \bigcup_{m \in F_n} \langle W_m \rangle, \quad n \in \mathbb{P}.$$

These non inclusions enables us to choose $f_n \in E(\{w_n\}) - \bigcup_{m \in F_n} \langle W_m \rangle$, $\| f_n \|_\infty = 1$, for each $n \in \mathbb{P}$. Then:

$$\{f_n : n \in \mathbb{P}\} \not\subseteq \langle W_m \rangle, \qquad \forall m \in \mathbb{N}$$

and the hypothesis $E \in \mathcal{B}_{ub}$ implies the existence of an unordered Baire-like space $(F, \tau)$, being $F \subset E$, $\{f_n : n \in \mathbb{P}\}$ bounded in $(F, \tau)$, and the topology $\tau$ is finer than the topology induced by $E$ in $F$. Therefore, there exists some $W_m$ containing a neighbourhood of zero in $(F, \tau)$. This implies the contradiction

$$\{f_n : n \in \mathbb{P}\} \subset \langle W_n \rangle$$

which proves the proposition. ■

## 3. Barrelledness of class $p$

Remember that a (Hausdorff) locally convex space $E$ is barrelled of class $p$ (or $p-$barrelled) if given a $p-$net $\mathcal{W} = \{E_t : t \in \mathbb{T}_p\}$ there is a $t \in \mathbb{N}^p$ such that $E_t$ is barrelled and dense in $E$. It is not difficult to see that when $E$ is $p-$barrelled there are many $E_t$, $t \in \mathbb{N}^p$, which are barrelled and dense in $E$, and the next definitions help us in obtaining the corresponding proof.

**Definition 1** Let $A$ be a subset of the set $\mathbb{N}$ of natural numbers. We will say that $A$ is a set of class $1$ *(*of strict class $1$*)* if $A$ is infinite *(*if there exists $n_1 \in \mathbb{N}$ such that $A = \{n \in \mathbb{N} : n \geq n_1\}$*)*.

A subset $A$ of $\mathbb{N}^p$ is a set of class $p$ *(*of strict class $p$*)* if $A = \bigcup_{b \in B_1} \{b\} \times C_b$, being $B_1$ a set of class $p-1$ *(*of strict class $p-1$*)* and such that each $C_b$ a set of class $1$ *(*of strict class $1$*)*.

It is obvious that a subset $A$ of class $p$ *(*of strict class $p$*)* may be written as $A = \bigcup_{b \in B_k} \{b\} \times C_b$, being $B_k$ a set of class $k$ *(*of strict class $k$*)* and each $C_b$ a set of class $p-k$ *(*of strict class $p-k$*)*, with $1 \leq k \leq p-1$.

Also an easy induction give us the next result.

**Proposition 3** *Let $A$ and $B$ be two sets of $\mathbb{N}^p$, then:*

1. *If $A$ is a set of strict class $p$ and $B$ is a set of strict class $p$ $($of class $p)$, then $A \cap B$ is of strict class $p$ $($of class $p)$.*

2. *$A$ contains a set of strict class $p$ if and only if $\mathbb{N}^p \setminus A$ does not contain a set of class $p$.*

3. *If $B$ is a set of class $p$ there exists a bijection $\varphi$ from $B$ onto $\mathbb{N}^p$ such that $\varphi$ preserves the lexicographic order.*

The last statement implies that if $W = \{E_t : t \in \mathbb{T}_p\}$ is a $p$-net in the $p-$barrelled space $E$ and $B$ is a set of class $p$ then there is a $t \in B$ such that $E_t$ is barrelled and dense in $E$, because the new numeration of $\{E_t : t \in B\}$ with the help of $\varphi$ gives a new $p-$net in $E$. From this observation the next proposition follows easily.

**Proposition 4** *A (Hausdorff) locally convex space is barrelled of class $p$ if and only if given a $p$-net* W$= \{E_t : t \in \mathbb{T}_p\}$ *there exists a set $A_p$ of strict class $p$ such that if $t \in A_p$ then $E_t$ is barrelled and dense in $E$.*

PROOF. Let $A = \{t \in \mathbb{N}^p : E_t$ is barrelled and dense in $E\}$. From the preceding observation it follows that $B = \mathbb{N}^p \setminus A$ cannot contain a set of class $p$. Then Proposition 3 statement 2, implies that $A$ contains a set of strict class $p$. ■

If $t = (t_1, t_2, \ldots, t_i, \ldots t_p) \in \mathbb{T}_p$ and $E_t$ is barrelled and dense in $E$ then, obviously, $E_{t_1 t_2 \ldots t_i}$ is barrelled and dense in $E$, for $1 \leq i \leq p-1$ ([6, Proposition 1.1.10]).

Recall that a locally convex space $E$ is Baire-like if given an increasing covering $\{A_n : n \in \mathbb{N}\}$ of $E$, being each $A_n$ a closed absolutely convex subset of $E$, there exists an $A_p$ which is a neighborhood of zero ([18]). It is obvious that suprabarrelled spaces are Baire-like, that Baire-like spaces are barrelled, that if $\{E_n : n \in \mathbb{N}\}$ is a linear increasing covering of the Baire-like space $E$ there exists an $E_n$ which is dense in $E$ and that if $F$ is barrelled and dense in the Baire space $E$ then $F$ is Baire-like ([6, Propositions 3.1.2 and 3.2.3]). Therefore a locally convex (Hausdorff) space $E$ is $p-$barrelled if given a $p$-net *W*$= \{E_t : t \in \mathbb{T}_p\}$ in $E$ it is verified one of the following conditions:

1. There exists $t \in \mathbb{N}^p$ such that $E_t$ is barrelled and dense in $E$.

2. There exists $t \in \mathbb{N}^p$ such that $E_t$ is Baire-like.

3. There is a set $A \subset \mathbb{N}^p$ of strict class $p$ such that for each $t \in A$ we have that $E_t$ is barrelled and dense in $E$.

4. There is a set $A \subset \mathbb{N}^p$ of strict class $p$ such that for each $t \in A$ we have that $E_t$ is Baire-like.

   In the two last conditions we may omit the word strict.

Now let us suppose that $\mathcal{W} = \{F_t : t \in \mathbb{T}_p\}$ is a $p$-net in $E$. Let $T_{n_1 n_2 \ldots n_p}$ be a barrel in $F_{n_1 n_2 \ldots n_p}$, $V_{n_1 n_2 \ldots n_p} = \overline{T_{n_1 n_2 \ldots n_p}}^E$, $Z_{n_1 n_2 \ldots n_p} = \langle V_{n_1 n_2 \ldots n_p} \rangle$, $S_{n_1 n_2 \ldots n_p} = \bigcap_{m=n_p}^{\infty} Z_{n_1 n_2 \ldots n_{p-1} m}$, $Z_{n_1 n_2 \ldots n_{p-1}} = \bigcup_{n_p=1}^{\infty} S_{n_1 n_2 \ldots n_p}$, $S_{n_1 n_2 \ldots n_{p-1}} = \bigcap_{m=n_{p-1}}^{\infty} Z_{n_1 n_2 \ldots n_{p-2} m}$, $\ldots$, $Z_{n_1} = \bigcup_{n_2=1}^{\infty} S_{n_1 n_2}$, $S_{n_1} = \bigcap_{m=n_1}^{\infty} Z_m$.

It is obvious that if $A$ is a set of strict class $p$ and $F \subset S_{n_1 n_2 \ldots n_p}$ for each $(n_1 n_2 \ldots n_p) \in A$ then we have that $F \subset S_{m_1}$, $F \subset S_{m_1 m_2}$, $\ldots$, $F \subset S_{m_1 m_2 \ldots m_{p-1}}$ and $F \subset S_{m_1 m_2 \ldots m_{p-1} m_p}$ when $(m_1 m_2 \ldots m_p) \in A$.

**Lemma 3** *Let $\mathcal{W} = \{F_t : t \in \mathbb{T}_p\}$ be a $p$-net in $E$ and let $T_{n_1 n_2 \ldots n_p}$ be a barrel in $F_{n_1 n_2 \ldots n_p}$. Suppose that given a sequence $\{f_p : p \in \mathbb{N}\} \in \mathcal{S}_E$ there exists a set $A$ of strict class $p$ such that $\{f_p : p \in \mathbb{N}\} \subset S_{n_1 n_2 \ldots n_p}$ for each $(n_1, n_2, \ldots, n_p) \in A$. Then there exists a countable set $\Delta$ (possibly empty) and a set $B$ of strict class $p$ such that $E(\Omega \setminus \Delta) \subset S_{n_1 n_2 \ldots n_p}$ for each $(n_1, n_2, \ldots, n_p) \in B$.*

PROOF. We are going to prove the lemma by decreasing induction. First we will see that there is a countable set $\Delta$ (possibly empty) and a natural number $n_1$ such that $E(\Omega \setminus \Delta) \subset S_{n_1}$.

In fact, if this were not true then we may find $f_1 \in E$, with $\|f_1\|_\infty = 1$ and $f_1 \notin S_1$. The set $\Delta_1 = supp(f_1)$ is countable and from $E(\Omega \setminus \Delta_1) \nsubseteq S_2$ we deduce the existence of a $f_2 \in E(\Omega \setminus \Delta_1)$ with $\|f_2\|_\infty = 1$ and $f_2 \notin S_2$. Then put $\Delta_2 = supp(f_2)$ and from $E(\Omega \setminus (\Delta_1 \cup \Delta_2)) \nsubseteq S_3$ we may suppose that there exists $f_3 \in E(\Omega \setminus (\Delta_1 \cup \Delta_2))$ with $\|f_3\|_\infty = 1$ and $f_3 \notin S_3$. Continuing in this way we determine by induction a unitary sequence $\{f_n : n \in \mathbb{N}\}$ in $E$ and a pairwise disjoint sequence $\{\Delta_n : n \in \mathbb{N}\}$ of countable subsets of $\Omega$ such that $\Delta_n = supp(f_n)$, $\|f_n\|_\infty = 1$ and $f_n \notin S_n$ , for $n = 1, 2, \ldots$

The sequence $\{f_q : q \in \mathbb{N}\} \in \mathcal{S}_E$. By hypothesis there exists a set $A$ of strict class $p$ such that $\{f_q : q \in \mathbb{N}\} \subset S_{n_1 n_2 \ldots n_p}$, for each $(n_1, n_2, \ldots, n_p) \in A$. We also have that $\{f_q : q \in \mathbb{N}\} \subset S_{n_1}$ if

$(n_1, n_2, \ldots, n_p) \in A$. From this inclusion follows the contradiction $f_{n_1} \in S_{n_1}$, which proves our assertion.

Continuing with the induction let us suppose that there exists a set $A_{h-1}$ of strict class $h-1$ and a countable set $\Delta$ such that $E(\Omega \setminus \Delta) \subset S_{n_1 n_2 \ldots n_{h-1}}$ for each $(n_1, n_2, \ldots, n_{h-1}) \in A_{h-1}$. Let $A_{h-1}^{'}$ and $A_{h-1}^{''}$ be a partition of $A_{h-1}$ such that:

• $A_{h-1}^{'}$ is formed by the elements $b = (n_1, n_2, n_3, \ldots, n_{h-1})$ belonging to $A_{h-1}$ for which we could determine a countable set $\Delta_b$ of $\Omega$ and a natural number $m$ such that $E(\Omega \setminus \{\Delta \cup \Delta_b\}) \subset S_{n_1 n_2 \ldots n_{h-1} m}$.

• $A_{h-1}^{''} = A_{h-1} \diagdown A_{h-1}^{'}$. If $(n_1, n_2, n_3, \ldots, n_{h-1}) \in A_{h-1}^{''}$, $\Delta^{''}$ is a countable subset of $\Omega$ and $m \in \mathbb{N}$ we have that $E\left(\Omega \setminus \left\{\Delta \cup \Delta^{''}\right\}\right) \nsubseteq S_{n_1 n_2 \ldots n_{h-1} m}$.

If $A_{h-1}^{'}$ contains a set $B$ of strict class $h-1$ then we immediately obtain the next step of the inductive process. In fact, if $\Delta^{'} = \bigcup_{b \in B} \Delta_b$ we have that for every $b \in B$ there exists a set $I_b$ of strict class 1 such that $E\left(\Omega \setminus \left\{\Delta \cup \Delta^{'}\right\}\right) \subset S_{n_1 n_2 \ldots n_{h-1} n_h}$ for every $(n_1, n_2, \ldots, n_{h-1}, n_h) \in \bigcup_{b \in B} \{b\} \times I_b = A_h$, being obvious that $A_h$ is a set of strict class $h$.

If $A_{h-1}^{'}$ does not contain a set of strict class $h-1$ then $A_{h-1}^{''}$ contains a set $B_{h-1}$ of class $h-1$ such that for each $(n_1, n_2, \ldots, n_{h-1}, n_h) \in B_{h-1} \times \mathbb{N}$ and each countable subset $\Delta^{''}$ of $\Omega$ we have that

$$E\left(\Omega \setminus \left\{\Delta \cup \Delta^{''}\right\}\right) \nsubseteq S_{n_1 n_2 \ldots n_{h-1} n_h} \tag{1}$$

It is obvious that $B_{h-1} \times \mathbb{N}$ is a set of class $h$, whose elements can be enumerated in the following way $\{(n_1(i), n_2(i), \ldots, n_{h-1}(i), n_h(i)) : i = 1, 2, 3, \ldots\}$.

From (1) we deduce that $E(\Omega \setminus \Delta) \nsubseteq S_{n_1(1), n_2(1), \ldots, n_h(1)}$ which enables us to determine $g_1 \in E(\Omega \setminus \Delta)$, with $\|g_1\|_\infty = 1$ and $g_1 \notin S_{n_1(1), n_2(1), \ldots, n_h(1)}$.

If $\Delta_1^{'} = supp(g_1)$ we have by (1) that $E\left(\Omega \setminus \left\{\Delta \cup \Delta_1^{'}\right\}\right) \nsubseteq S_{n_1(2), n_2(2), \ldots, n_h(2)}$. This relation indicates the existence of $g_2 \in E\left(\Omega \setminus \left\{\Delta \cup \Delta_1^{'}\right\}\right)$, with $\|g_2\|_\infty = 1$ and $g_2 \notin S_{n_1(2), n_2(2), \ldots, n_h(2)}$.

Now, if $\Delta_2^{'} = supp(g_2)$, we also have by (1) that $E\left(\Omega \setminus \left\{\Delta \cup \Delta_1^{'} \cup \Delta_2^{'}\right\}\right) \nsubseteq S_{n_1(3), n_2(3), \ldots, n_h(3)}$.

Therefore, and after an obvious induction, we could obtain a sequence $\{g_i : i \in \mathbb{N}\}$ of unitary vectors in $E$ with pairwise disjoint supports $\Delta_i^{'} = supp(g_i)$, $i = 1, 2, 3, \ldots$, such that $g_i \notin S_{n_1(i), n_2(i), \ldots, n_h(i)}$, $i = 1, 2, 3, \ldots$.

The sequence $\{g_i : i \in \mathbb{N}\} \in \mathcal{S}_E$ and therefore there exists a set $C$ of strict class $p$ such that $\{g_i : i \in \mathbb{N}\} \subset S_{n_1 n_2 \ldots n_h}$ for $(n_1, n_2, \ldots, n_h, n_{h+1}, \ldots, n_p) \in C$. By Proposition 3 statement 1 there exists an index $k$ such that $\{g_i : i \in \mathbb{N}\} \subset S_{n_1(k), n_2(k), \ldots, n_h(k)}$.

This relation contains the contradiction $g_k \in S_{n_1(k), n_2(k), \ldots, n_h(k)}$ that proves this lemma. ■

**Lemma 4** *Let $\mathcal{W} = \{F_t : t \in \mathbb{T}_p\}$ be a $p$-net in the (Hausdorff) locally convex space $E$ and let $T_{n_1 n_2 \ldots n_p}$ be a barrel in $F_{n_1 n_2 \ldots n_p}$. If $F$ is a $p$-barrelled subspace of $E$ then there exists a subset $A$ of strict class $p$ such that $F \subset S_{n_1 n_2 \ldots n_p}$ whenever $(n_1 n_2 \ldots n_p) \in A$.*

PROOF. Since $\{F \cap F_t : t \in \mathbb{T}_p\}$ is a $p$-net in $F$, there is a set $A$ of strict class $p$ such that if $(n_1 n_2 \ldots n_p) \in A$ then $F \cap F_{n_1 n_2 \ldots n_p}$ is barrelled and dense in $F$. By density, $\overline{F \cap T_{n_1 n_2 \ldots n_p}}^F$ is a neighborhood of zero in $F$. Therefore $F \subset Z_{n_1 n_2 \ldots n_p}$ for every $(n_1 n_2 \ldots n_p) \in A$, which implies that $F \subset S_{n_1 n_2 \ldots n_p}$ whenever $(n_1 n_2 \ldots n_p) \in A$. ■

**Lemma 5** *Let $\mathcal{W} = \{F_t : t \in \mathbb{T}_p\}$ be a $p$-net in the (Hausdorff) locally convex space $E$, let $T_{n_1 n_2 \ldots n_p}$ be a barrel in $F_{n_1 n_2 \ldots n_p}$, $F$ a subspace of $E$ and $\tau$ a locally convex topology in $F$ finer than the induced by $E$. If $(F, \tau)$ is $p$-barrelled then there exists a set $A$ of strict class $p$ such that $F \subset S_{n_1 n_2 \ldots n_p}$ whenever $(n_1 n_2 \ldots n_p) \in A$.*

PROOF. Since $(F,\tau)$ is $p$-barrelled and $\{F\cap F_t : t\in\mathbb{T}_p\}$ is a $p$-net in $F$, there is a set $A$ of strict class $p$ such that if $(n_1n_2\ldots n_p)\in A$ then $F\cap F_{n_1n_2\ldots n_p}$ is barrelled and dense in $(F,\tau)$. Hence $F\cap T_{n_1n_2\ldots n_p}$ is a neigbourhood of zero in $F\cap F_{n_1n_2\ldots n_p}$ endowed with the topology induced by $\tau$ and, by density, $\overline{F\cap T_{n_1n_2\ldots n_p}}^{(F,\tau)}$ is a neighborhood of zero in $(F,\tau)$. ¿From $\overline{F\cap T_{n_1n_2\ldots n_p}}^{(F,\tau)}\subset\overline{T_{n_1n_2\ldots n_p}}^{E}$ it follows that $F\subset Z_{n_1n_2\ldots n_p}$ for every $(n_1n_2\ldots n_p)\in A$, which implies that $F\subset S_{n_1n_2\ldots n_p}$whenever $(n_1n_2\ldots n_p)\in A$. ■

Now, in the family $\mathcal{B}$ we are going to consider the subfamily $\mathcal{B}_p$ such that $E\in\mathcal{B}$ belongs to $\mathcal{B}_p$ if given the sequence $\{f_n : n\in\mathbb{N}\}\in\mathcal{S}_E$ there exists a $p-$barrelled space $(F,\tau)$ such that $\{f_n : n\in\mathbb{N}\}\subset F\subset E$ and $\tau$ is a locally convex topology finer than the topology induced in $F$ by the topology of $E$.

**Lemma 6** *Let $\mathcal{W}=\{F_t : t\in\mathbb{T}_p\}$ be a $p$-net in $E$ and let $T_{n_1n_2\ldots n_p}$ be a barrel in $F_{n_1n_2\ldots n_p}$. If $E\in\mathcal{B}_p$ and $X$ is a $p-$barrelled space then there exists a set $A$ of strict class $p$ such that $E=S_{n_1n_2\ldots n_p}$ when $(n_1,n_2,\ldots,n_p)\in A$.*

PROOF. As $E\in\mathcal{B}_p$ Lemma 5 guaranties that we may apply Lemma 3. Therefore it is enough to prove this lemma when $\Omega$ is a countable set. Therefore we are going to suppose that $\Omega=\mathbb{N}$ and we will obtain the proof by decreasing induction.

Now we are going to prove that there exists a natural number $n_1$ such that $E=S_{n_1}$. In first place, we will find a natural number $i$ such that $E(\{i+1,i+2,\ldots\})\subset S_i$. In fact if this were not possible we would determine a sequence $\{f_i : i\in\mathbb{N}\}$ of unitary vectors such that $f_i\in E\left(\{i+1,i+2,\ldots\}\right)\setminus S_i$, for $i=1,2,\ldots$

The relation $\{f_i : i\in\mathbb{N}\}\in\mathcal{S}_E$, $E\in\mathcal{B}_p$ and Lemma 5 implies that there exists a set $A$ of strict class $p$ such that $\{f_i : i\in\mathbb{N}\}\in S_{n_1n_2\ldots n_p}$ when $(n_1,n_2,\ldots,n_p)\in A$. Therefore if $(n_1,n_2,\ldots,n_p)\in A$ we have that $\{f_i : i\in\mathbb{N}\}\in S_{n_1}$ implying the contradiction $f_{n_1}\in S_{n_1}$.

This enables us to suppose that there exists a natural number $i$ such that $E(\{i+1,i+2,\ldots\})\subset S_i$. Let $\Delta=\{1,2,\ldots,i\}$.

Since the space $E\left(\Delta\right)$ is isometric to $X^{\Delta}$ endowed with the $l_\infty$ norm, we have that $E\left(\{1,2,\ldots,i\}\right)$ is $p$-barrelled (see [6, Proposition 2.3]). Hence, there is a set $B$ of strict class $p$ such that $X^{\Delta}\subset S_{n_1n_2\ldots n_p}$ when $(n_1,n_2,\ldots,n_p)\in B$ (Lemma 4). Then for each $(m_1,m_2,\ldots,m_p)\in B$ we have that $X^{\Delta}\subset S_{m_1}$. If $n_1\geq\max(i,m_1)$ then $E=E(\mathbb{N})=E(\{1,2,\ldots,i\})+E\left(\{i+1,i+2,\ldots\}\right)\subset S_i+S_{m_1}\subset S_{n_1}+S_{n_1}\subset S_{n_1}$

Let us suppose that in the $(h-1)$-step of the inductive process we have determined a set $A_{h-1}$ of class $h-1$ such that $E\left(\mathbb{N}\right)=S_{n_1n_2\ldots n_{h-1}}$ whenever $(n_1,n_2,\ldots,n_{h-1})\in A_{h-1}$. Let $b=(n_1,n_2,\ldots,n_{h-1})\in A_{h-1}$. The sets $S_{n_1n_2\ldots n_h\ldots n_i\ldots n_s}$, with $n_i\in\mathbb{N}$, $h\leq i\leq s\leq p$ generate in a natural way the $p-(h-1)$-net formed by the sets $S'_{n_1n_2\ldots n_h\ldots n_i\ldots n_s}$, with $n_i\in\mathbb{N}$ for $h\leq i\leq s\leq p$, given by

$$S'_{n_1n_2\ldots n_h}=S_{n_1n_2\ldots n_h}$$

and

$$S'_{n_1n_2\ldots n_h\ldots n_i\ldots n_{s-1}n_s}=S_{n_1n_2\ldots n_h\ldots n_i\ldots n_{s-1}n_s}\cap S'_{n_1n_2\ldots n_h\ldots n_i\ldots n_{s-1}}$$

with $n_i\in\mathbb{N}$, $h\leq i\leq s\leq p$ and $h\leq s-1$.

In $S'_{n_1n_2\ldots n_p}$ consider the barrel $T'_{n_1n_2\ldots n_p}=\overline{T_{n_1n_2\ldots n_p}}^{E(\mathbb{N})}\bigcap S'_{n_1n_2\ldots n_p}$, being $T_{n_1n_2\ldots n_p}$ the barrel given in $F_{n_1n_2\ldots n_p}(\subset S'_{n_1n_2\ldots n_p}\subset S_{n_1n_2\ldots n_p})$. Since $\overline{T'_{n_1n_2\ldots n_p}}^{E(\mathbb{N})}=\overline{T_{n_1n_2\ldots n_p}}^{E(\mathbb{N})}$, applying the first step of the inductive process to the $p-(h-1)$-net

$$\left\{S'_{n_1n_2\ldots n_h\ldots n_i\ldots n_s} : n_i\in\mathbb{N}\text{ y } h\leq i\leq s\leq p\right\}$$

we get a set $I_b$ of class 1 such that $E\left(\mathbb{N}\right)=S_{n_1n_2\ldots n_{h-1}n_h}$ for each $n_h\in I_b$. The induction finishes with the observation that $\bigcup_{b\in A_{h-1}}\{b\}\times I_b$ is a set $A_h$ of strict class $h$ and that $E\left(\mathbb{N}\right)=S_{n_1n_2\ldots n_{h-1}n_h}$ when $(n_1,n_2,\ldots,n_h)\in A_h$. This induction proves the lemma. ■

**Theorem 1** *Let $\Omega$ be a non void set and suppose that $E \in \mathcal{B}_p$. The locally convex space $E$ is $p$-barrelled if and only if $X$ is $p$-barrelled.*

PROOF. For $\omega \in \Omega$ we have that $E$ and the product $E(\Omega \diagdown \{\omega\}) \times E(\{\omega\})$ with the supremum norm are isometric. Therefore, the spaces $E(\{\omega\})$, and $E \diagup E(\Omega \diagdown \{\omega\})$ are isometric. Hence if $E$ is $p$-barrelled, then $X$ (isometric to $E(\{\omega\})$) is $p$-barrelled according to [6, Proposition 3.2.12].

Conversely, if $X$ is $p$-barrelled we are going to show by induction that $E$ is $p$-barrelled.

Now we prove that $E$ is suprabarrelled (1-barrelled). We know that $E$ is Baire-like (see Proposition 1 and [6, Proposition 1.2.1]). Therefore if $\{F_n : n \in \mathbb{N}\}$ is a 1-net in $E$ there is a $n$ such that $F_m$ is dense in $E$ for $m \geq n$. If $E$ were not suprabarrelled we would find in $E$ a 1-net $\{F_n : n \in \mathbb{N}\}$ such that each $F_n$ is dense and non barrelled. Let $T_n$ be a barrel in $F_n$ which is not neighborhood of zero in $F_n$. Set $V_n = \overline{T_n}^E$ and $S_n = \bigcap_{m \geq n} \langle V_n \rangle$. According to Lemma 6, there is some $S_n = E$, hence $E = \langle V_n \rangle$ and the barrelledness of $E$ ( Proposition 1) yields that $V_n$ is a neighborhood of zero in $E$. Then $T_n = V_n \cap F_n$ is a neighborhood of zero in $F_n$, a contradiction that enables us to establish that there exists a $F_n$ which is barrelled.

Assuming that $E$ is $(p-1)$-barrelled, $p \geq 2$, and that $\{F_t : t \in \mathbb{T}_p\}$ is a $p$-net in $E$, then we can suppose that there is a set $A_{p-1}$ of class $(p-1)$ such that $F_t$ is barrelled and dense in $E$ for $t \in A_{p-1}$. If $t \in A_{p-1}$ then $F_t$ is Baire-like ([6, Proposition 1.2.1]), hence there is a set $A_p$ of class $p$ such that $F_{tn_p}$ is dense in $E$ for $(t, n_p) \in A_p$. Consequently, if $E$ were not $p$-barrelled we may find a $p$-net $\left\{F_t : t \in \mathbb{T}'_p\right\}$ such that each $F_t$, for $t \in \mathbb{N}^p$, is not barrelled and dense in $E$. Let $T_t$ be a barrel in $F_t$ for $t \in \mathbb{N}^p$, which is not neighborhood of zero in $F_t$, for $t \in \mathbb{N}^p$. According to Lemma 6, there is a $(n_1, n_2, \ldots, n_p)$ such that $S_{n_1 n_2 \ldots n_p} = E$ and then, by barrelledness (Proposition 1), $V_{n_1 n_2 \ldots n_{p-1} n_p} = \overline{T_{n_1 n_2 \ldots n_{p-1} n_p}}^E$ is a neighborhood of zero in $S_{n_1 n_2 \ldots n_{p-1} n_p}$. This implies the contradiction that $T_{n_1 n_2 \ldots n_{p-1} n_p} = V_{n_1 n_2 \ldots n_{p-1} n_p} \cap F_{n_1 n_2 \ldots n_{p-1} n_p}$ is a zero neighborhood in $F_{n_1 n_2 \ldots n_{p-1} n_p}$. Hence $E$ is $p$-barrelled. ■

If we apply Lemma 6 when $T_{n_1 n_2 \ldots n_p} = F_{n_1 n_2 \ldots n_p}$ we obtain the following property.

Let $\mathcal{W} = \{F_t : t \in \mathbb{T}_p\}$ be a $p-$net in $E$. If $E \in \mathcal{B}_p$ and $X$ is a $p-$barrelled space, then there exists a set $A$ of strict class $p$ such that $F_t$ is dense in $E$ when $t \in A$.

This property simplifies the second part of the proof of Theorem 1. In fact, if $X$ is $p-$barrelled and $E$ were not $p-$barrelled we may find a $p-$net $\{F_t : t \in \mathbb{T}_p\}$ such that $F_t$ is not barrelled and dense in $E$, for each $t \in \mathbb{N}^p$. We obtain the same contradiction as in the end of the proof of Theorem 1.

## 4. Notes

When $E$ is $c_0(\Omega, X)$ or $bcs(\Omega, X)$ and $\{f_n : n \in \mathbb{N}\}$ is a sequence of unitary vectors with disjoint supports, it is easy to prove that $\{\sum_{n=1}^{\infty} \alpha_n f_n : |\alpha_n| \leq 1, \ \ n = 1, 2, \ldots\}$ is a Banach disk.

If $\{f_n : n \in \mathbb{N}\}$ is a sequence of unitary vectors of $E$ and there is a countable set $\{w_1, w_2, \ldots\} \subset \Omega$, such that $supp(f_n) \subset \{w_{n+1}, w_{n+2}, \ldots\}$ then it is obvious that $\{\sum_{n=1}^{\infty} \alpha_n f_n : \sum_{n=1}^{\infty} |\alpha_n| \leq 1\}$ is a Banach disk.

Therefore, $c_0(\Omega, X)$ and $bcs(\Omega, X)$ are $p-$barrelled if and only if $X$ is $p-$barrelled.

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