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Fréchet-space-valued measures and the AL-property

S. Okada and W. J. Ricker

Dedicated to the late K. Floret

Abstract. Associated with every vector measure m taking its values in a Fréchet space X is the space $L^1(m)$ of all m-integrable functions. It turns out that $L^1(m)$ is always a Fréchet lattice. We show that possession of the AL-property for the lattice $L^1(m)$ has some remarkable consequences for both the underlying Fréchet space X and the integration operator $f \mapsto \int f\, dm$.

Medidas con valores en espacios de Fréchet y la propiedad AL

Resumen. Cada medida vectorial m con valores en un espacio de Fréchet X tiene asociado el espacio $L^1(m)$ de todas las funciones m-integrables. Este espacio resulta ser siempre un retículo de Fréchet. Demostramos que cuando el retículo $L^1(m)$ goza de la propiedad AL, se obtienem consecuencias notables tanto para el espacio de Fréchet X como para el operador integración $f \mapsto \int f\, dm$.

1. Introduction and main results

Associated to every vector measure m, with values in some Banach space X, is the Banach lattice $L^1(m)$ consisting of the space of all m-integrable functions equipped with the topology of convergence in mean. "How close" are such spaces $L^1(m)$ to being classical L^1-spaces (corresponding to some positive measure)? Due to a fundamental result of S. Kakutani, [16, Theorem 1.b.2], this question can be reformulated: when is $L^1(m)$ order isomorphic to an AL-space, that is, to a Banach lattice in which the norm is additive on the positive cone? This question was completely and satisfactorily answered by G. Curbera in [5, Proposition 3.1] and [6, Proposition 2], where an important connection is made between the AL-property of the lattice $L^1(m)$ and certain properties of the *integration operator* $I_m : L^1(m) \to X$ given by $f \mapsto \int f\, dm$. Curbera's result was subsequently extended to the setting of (locally convex) Fréchet spaces X by A. Fernández and F. Naranjo, [9, Theorem 2.1]. The aim of this note is to show that possession of the AL-property for the Fréchet lattice $L^1(m)$ has some remarkable consequences for both the underlying Fréchet space X and the integration operator I_m.

Theorem 1 *Let X be a Fréchet space. Then X is nuclear if and only if $L^1(m)$ is a Fréchet AL-lattice for every X-valued vector measure m.* $\square$

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Some notation is needed to formulate the next result. Let X be a (complex) Fréchet space with continuous dual space X'. It is always assumed that $p_1 \leq p_2 \leq \ldots$ is an increasing sequence of continuous seminorms determining the topology of X. Let $X/p_k^{-1}(\{0\})$ be the *quotient normed space* determined by p_k and X_k denote its Banach space completion, for $k \in \mathbb{N}$. The norm in X_k is denoted by $\|\cdot\|_k$ and the *canonical quotient map* of X onto $X/p_k^{-1}(\{0\})$ is denoted by Π_k; we use the same notation Π_k when it is interpreted as being X_k-valued.

Let Σ be a σ-algebra of subsets of a non-empty set Ω and $m : \Sigma \to X$ be a *vector measure*, i.e., m is σ-additive meaning that $m(E_n) \to 0$ (in X) whenever $E_n \downarrow \emptyset$ in Σ. The continuity of Π_k ensures that $m_k := \Pi_k \circ m$ is a vector measure on Σ with values in $X/p_k^{-1}(\{0\}) \hookrightarrow X_k$, for $k \in \mathbb{N}$. For the definition of the *variation measure* $|m_k| : \Sigma \to [\,0, \infty\,]$ of the Banach-space-valued measure $m_k : \Sigma \to X_k$ we refer to [7, pp. 2–3].

Theorem 2 *Let X be a Fréchet Montel space and m be an X-valued vector measure. The following statements are equivalent.*

(i) *The Fréchet lattice $L^1(m)$ is isomorphic (orderwise and topologically) to a Banach AL-lattice.*

(ii) *There exists $r \in \mathbb{N}$ such that the Fréchet lattice $L^1(m)$ is isomorphic to the Banach lattice $L^1(|m_r|)$.*

(iii) *The Fréchet space $L^1(m)$ is normable.*

(iv) *The integration operator $I_m : L^1(m) \to X$ is compact.* ∎

Recall that a Fréchet space is *Montel* if every closed bounded subset is compact. Every *nuclear* Fréchet space is necessarily Montel, [22, p. 520], but not conversely, [13, pp. 433–434]. It should be pointed out, in relation to Theorem 1, that other characterizations of nuclear Fréchet spaces in terms of the variation of vector measures, [8], [15], and certain properties of the range of vector measures, [4], are also known.

2. Preliminaries

In this section we formulate the necessary lemmata (and some results of interest in their own right) which are needed to establish Theorem 1 and Theorem 2. The notation is as in Section 1. The following fact is well known; see [13, Ch. 3, Section 4].

Lemma 1 *Let X be a Fréchet space with topology determined by an increasing sequence of continuous seminorms $p_1 \leq p_2 \leq \ldots$. Then, for each $k \in \mathbb{N}$, we have $\|\Pi_k(x)\|_k = p_k(x)$ for $x \in X$.* ∎

An immediate consequence of Lemma 1 is, for any vector measure $m : \Sigma \to X$, that

$$|m_k|(E) = \sup \left\{ \sum_j p_k\big(m(E_j)\big) : \{E_j\} \in \mathcal{P}(E) \right\}, \qquad E \in \Sigma, \tag{1}$$

for each $k \in \mathbb{N}$, where $\mathcal{P}(E)$ is the set of all finite partitions of E via Σ-measurable sets E_j. We say that m has *finite variation* if *every* Banach-space-valued measure m_k, for $k \in \mathbb{N}$, has finite variation (i.e., $|m_k|(\Omega) < \infty$). Because of (1) this agrees with the definition given in [3, p. 336], where it is called "bounded variation".

Let (Ω, Σ) be a measurable space, X be a Fréchet space (with topology given by continuous seminorms $p_1 \leq p_2 \leq \ldots$) and $m : \Sigma \to X$ be a vector measure. For $x' \in X'$, let $\langle m, x' \rangle$ denote the complex measure $E \mapsto \langle m(E), x' \rangle$; its variation $|\langle m, x' \rangle|$ is then a finite measure. A Σ-measurable function $f : \Omega \to \mathbb{C}$ is m-*integrable* if it is $\langle m, x' \rangle$-integrable for each $x' \in X'$, and if there is a set function $fm : \Sigma \to X$ satisfying $\langle (fm)(E), x' \rangle = \int_E f \, d\langle m, x' \rangle$ for all $x' \in X'$ and $E \in \Sigma$. The classical notation $\int_E f \, dm := (fm)(E)$

is also used. By the *Orlicz-Pettis theorem* (see [11, p. 308] or [12, p. 4]), fm is also a vector measure. The vector space of all individual m-integrable functions is denoted by $\mathcal{L}^1(m)$. For $k \in \mathbb{N}$, define a seminorm on $\mathcal{L}^1(m)$ by

$$p_k(m) : f \mapsto \sup\left\{p_k\left(\int_E f\,dm\right) : E \in \Sigma\right\}, \qquad f \in \mathcal{L}^1(m). \tag{2}$$

Equipped with the seminorms $p_1(m) \leq p_2(m) \leq \ldots$ the locally convex space $\mathcal{L}^1(m)$ is complete and contains the space of all $\mathbb{C}$-valued, Σ-*simple functions* (which is denoted by $\mathrm{sim}(\Sigma)$) as a dense subspace; see [12, Ch. IV] and [14, Theorem 2.4], or also [10]. Continuity of the integration operator $I_m : \mathcal{L}^1(m) \to X$ (see Section 1) follows from the inequalities

$$p_k(I_m\, f) \leq p_k(m)(f), \qquad f \in \mathcal{L}^1(m), \quad k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, the *polar* of the p_k-unit ball $U_k := \{x \in X : p_k(x) \leq 1\}$ is defined by

$$U_k^\circ := \big\{x' \in X' : |\langle x, x'\rangle| \leq 1 \text{ for all } x \in U_k\big\}.$$

The family of seminorms

$$p_k[m](f) := \sup\left\{\int_\Omega |f|\, d|\langle m, x'\rangle| : x' \in U_k^\circ\right\}, \qquad f \in \mathcal{L}^1(m), \tag{3}$$

for $k \in \mathbb{N}$, is equivalent to the seminorms in (2) since

$$p_k(m)(f) \leq p_k[m](f) \leq 4p_k(m)(f), \qquad f \in \mathcal{L}^1(m), \tag{4}$$

for each $k \in \mathbb{N}$; see [12, Lemma II.1.2], where there is 2 in place of 4 above because X is considered over $\mathbb{R}$ rather than $\mathbb{C}$. With respect to the positive cone $\{f \in \mathcal{L}^1(m) : f \geq 0\}$, defined for the pointwise order on Ω, the space $\mathcal{L}^1(m)$ becomes a countably seminormed lattice with respect to the *lattice seminorms* (3).

Lemma 2 *Let X be a Fréchet space and $m : \Sigma \to X$ be a vector measure. Then*

$$\bigcap_{k=1}^{\infty} \mathcal{L}^1(|m_k|) \subseteq \mathcal{L}^1(m) = \bigcap_{k=1}^{\infty} \mathcal{L}^1(m_k). \tag{5}$$

Moreover, the inclusion is continuous when $\bigcap_{k=1}^{\infty} \mathcal{L}^1(|m_k|)$ is equipped with the topology given by the increasing sequence of seminorms

$$|||f|||_k := \int_\Omega |f|\, d|m_k|, \qquad k \in \mathbb{N}. \tag{6}$$

In addition, the topology of $\mathcal{L}^1(m)$ is equivalent to that given by the increasing sequence of seminorms

$$\|\cdot\|_k(m_k)(f) := \sup\left\{\left\|\int_E f\,dm_k\right\|_k : E \in \Sigma\right\}, \qquad f \in \mathcal{L}^1(m), \quad k \in \mathbb{N}. \tag{7}$$

PROOF. All of the statements in the Lemma are contained in [18, Lemma 2.4] except for the equality in (5), where it is only established that $\mathcal{L}^1(m) \subseteq \bigcap_{k=1}^{\infty} \mathcal{L}^1(m_k)$.

For the converse inclusion, let $f \in \bigcap_{k=1}^{\infty} \mathcal{L}^1(m_k)$. Choose a sequence $\{s_n\}_{n=1}^{\infty}$ in $\mathrm{sim}(\Sigma)$ which converges pointwise to f on Ω and satisfies $|s_n| \leq |f|$ for all $n \in \mathbb{N}$. To verify that $f \in \mathcal{L}^1(m)$ it suffices to show, given $E \in \Sigma$, that $\{\int_E s_n\, dm\}_{n=1}^{\infty}$ is Cauchy in X, [14, Theorem 2.4]. So, fix $k \in \mathbb{N}$. Then, by Lemma 1 we have

$$\begin{aligned} p_k\left(\int_E (s_n - s_r)\, dm\right) &= \left\|\Pi_k\left(\int_E (s_n - s_r)\, dm\right)\right\|_k \\ = \left\|\int_E (s_n - s_r)\, d\,(\Pi_k \circ m)\right\|_k &= \left\|\int_E (s_n - s_r)\, dm_k\right\|_k \longrightarrow 0 \end{aligned}$$

as $n, r \to \infty$ because of the dominated convergence theorem for vector measures, [12, Theorem II.4.2], applied to the Banach-space-valued measure m_k, i.e., $\lim_{n\to\infty} \int_E s_n \, dm_k = \int_E f \, dm_k$ in X_k. ■

A Fréchet lattice X has the AL-property, [23], if its topology can be defined by a sequence of continuous lattice seminorms $p_1, p_2, \ldots$ that are additive in the positive cone X^+, that is, such that

$$p_k(x+y) = p_k(x) + p_k(y), \qquad x, y \in X^+, \quad k \in \mathbb{N}.$$

It is a consequence of Kantorovič's theorem, [1, Theorem 1.7], that a Fréchet lattice is an AL-space if and only if equicontinuous subsets of X' are order bounded in X'.

Let X be a Fréchet space and $m : \Sigma \to X$ be a vector measure. For $k \in \mathbb{N}$, the variation measure $|m_k|$ of $m_k : \Sigma \to X_k$ has already been defined. Then $p_1 \leq p_2 \leq \ldots$ implies that $|m_1| \leq |m_2| \leq \ldots$ on Σ. An element $f \in \mathcal{L}^1(m)$ is called m-*null* if fm is the zero vector measure. Equivalently, $p_k(m)(f) = 0$ for all $k \in \mathbb{N}$, which in turn is equivalent to f being $|m_k|$-null for every $k \in \mathbb{N}$; this is based on the inclusions $\mathcal{L}^1(m) \subseteq \mathcal{L}^1(m_k)$ which follow from $m_k := \Pi_k \circ m$ (see Section 2 of [18]). The closed subspace of $\mathcal{L}^1(m)$ consisting of all m-null functions is denoted by $\mathcal{N}(m)$. Of course, $\mathcal{N}(m) = \bigcap_{k=1}^{\infty} \mathcal{N}(m_k) = \bigcap_{k=1}^{\infty} \mathcal{N}(|m_k|)$. The quotient space $L^1(m) := \mathcal{L}^1(m)/\mathcal{N}(m)$ becomes a Fréchet lattice when equipped with the quotient topology determined by the seminorms (3). Unlike in Banach spaces, where m and $|m|$ have the same null sets, for a Fréchet-space-valued measure m there is no single positive measure which plays the role of the variation. It follows from Lemma 2 that

$$\left(\bigcap_{k=1}^{\infty} \mathcal{L}^1(|m_k|) \right) \Big/ \mathcal{N}(m) \subseteq L^1(m) = \left(\bigcap_{k=1}^{\infty} \mathcal{L}^1(m_k) \right) \Big/ \mathcal{N}(m), \tag{8}$$

with a continuous inclusion. We adopt the notation of [9] and denote the left-hand-side of (8) by $L^1(|m|)$, even though the symbol $|m|$ has no meaning by itself if X is not normable. So, $L^1(|m|)$ is continuously included in $L^1(m)$ and both spaces have the same order. Furthermore, $L^1(|m|)$ is a Fréchet lattice which has the AL-property for the (induced quotient) seminorms (6); see [9]. The following result is (part of) Theorem 2.1 in [9], where it is proved for spaces over $\mathbb{R}$; the extension from $\mathbb{R}$ to $\mathbb{C}$ is straightforward. The Banach space version is due to G. Curbera, [6, Proposition 2].

Lemma 3 *Let X be a Fréchet space and m be an X-valued vector measure. Then the Fréchet lattice $L^1(m)$ is (orderwise and topologically) isomorphic to a Fréchet AL-lattice if and only if the natural inclusion $J : L^1(|m|) \to L^1(m)$ is a bicontinuous lattice isomorphism of $L^1(|m|)$ onto $L^1(m)$.* ■

The next result is of interest in its own right.

Proposition 1 *Let X be a Fréchet space and m be an X-valued vector measure. Then the Fréchet AL-lattice $L^1(|m|)$ is normable if and only if there exists $r \in \mathbb{N}$ such that $L^1(|m|)$ is orderwise and topologically isomorphic to the Banach AL-lattice $L^1(|m_r|)$.*

PROOF. Suppose that $L^1(|m|)$ is normable. Let $\Pi : \mathcal{L}^1(|m|) \to L^1(|m|)$ denote the canonical quotient map of $\mathcal{L}^1(|m|) := \bigcap_{k=1}^{\infty} \mathcal{L}^1(|m_k|)$ onto $L^1(|m|)$. If Q is a norm determining the topology of $L^1(|m|)$, then $q := Q \circ \Pi$ is a seminorm determining the topology of $\mathcal{L}^1(|m|)$. In other words, the following two statements hold:

(a) Given $k \in \mathbb{N}$, there is $\alpha_k > 0$ such that

$$|||f|||_k \leq \alpha_k \, q(f), \qquad f \in \mathcal{L}^1(|m|);$$

and

(b) There exist $r \in \mathbb{N}$ and $\beta_r > 0$ such that

$$q(f) \leq \beta_r |||f|||_r, \qquad f \in \mathcal{L}^1(|m|).$$

Let $k \in \mathbb{N}$ satisfy $k \geq r$. Then

$$|m_r|(E) \leq |m_k|(E) \leq \alpha_k \, \beta_r |m_r|(E), \qquad E \in \Sigma. \tag{9}$$

In fact, the first inequality follows because $p_r \leq p_k$ implies $|m_r| \leq |m_k|$ on Σ. By (a) and (b) above we have

$$|m_k|(E) \leq \alpha_k \, q(\chi_E) \leq \alpha_k \, \beta_r |m_r|(E)$$

so that (9) holds.

An immediate consequence of (9) is that $\mathcal{L}^1(|m_r|) = \mathcal{L}^1(|m_k|)$ for every $k \geq r$ and hence, that

$$\mathcal{L}^1(|m|) = \bigcap_{k=1}^{\infty} \mathcal{L}^1(|m_k|) = \mathcal{L}^1(|m_r|)$$

because $\mathcal{L}^1(|m_1|) \supseteq \mathcal{L}^1(|m_2|) \supseteq \ldots \supseteq \mathcal{L}^1(|m_r|)$. Similarly, (9) implies that $\mathcal{N}(m) = \bigcap_{k=1}^{\infty} \mathcal{N}(m_k) = \mathcal{N}(m_r)$ after recalling that $\mathcal{N}(m_j) = \mathcal{N}(|m_j|)$ for each $j \in \mathbb{N}$. So, $L^1(|m|) = L^1(|m_r|)$ as ordered vector spaces. From the definition of the topology on $L^1(|m|)$ it follows that the identity map from the Fréchet space $L^1(|m|)$ onto the Banach space $L^1(|m_r|)$ is continuous. This map is then an isomorphism by the *open mapping theorem*, [22, p. 172], and hence, $L^1(|m|)$ is isomorphic to the Banach AL-lattice $L^1(|m_r|)$.

The converse statement is obvious. ■

Corollary 1 *Let X be a Fréchet space and m be an X-valued vector measure. Then the Fréchet lattice $L^1(m)$ is a Banach AL-lattice if and only if there exists $r \in \mathbb{N}$ such that $L^1(m)$ is isomorphic to $L^1(|m_r|)$.*

PROOF. If $L^1(m)$ is a Banach AL-lattice, then it is also a Fréchet AL-lattice and so, by Lemma 3, $L^1(m) = L^1(|m|)$ with equality meaning orderwise and topologically isomorphic. But, $L^1(m)$ is also normable and so $L^1(|m|) = L^1(|m_r|)$ for some $r \in \mathbb{N}$ (see Proposition 1). Hence, $L^1(m)$ is isomorphic to $L^1(|m_r|)$.

The converse statement is obvious. ■

Recall that a continuous linear map T from a locally convex space Y into a Fréchet space X is *compact* if there is a neighbourhood U of $0 \in Y$ such that the closure of $T(U)$ is compact in X, [22, p. 483].

The sequence space $\omega := \mathbb{C}^{\mathbb{N}}$ is defined in Section 3; see Remark 1.

Proposition 2 *Let X be a Fréchet space and m be an X-valued vector measure whose integration operator $I_m : L^1(m) \to X$ is compact. Then $L^1(m)$ is a Banach AL-lattice. In particular, $L^1(m)$ cannot contain an isomorphic copy of the Fréchet space ω.*

PROOF. By [18, Theorem 2] there exists $r \in \mathbb{N}$ such that $I_{m_k} : L^1(m_k) \to X_k$ is compact and $L^1(m_k) = L^1(m_r)$ for all $k \geq r$. In particular,

$$\mathcal{L}^1(m_k) = \mathcal{L}^1(m_r) \quad \text{and} \quad \mathcal{N}(m_k) = \mathcal{N}(m_r), \qquad k \geq r. \tag{10}$$

Since $\mathcal{N}(m_1) \supseteq \mathcal{N}(m_2) \supseteq \ldots$ we conclude that $\mathcal{N}(m) = \bigcap_{k=1}^{\infty} \mathcal{N}(m_k) = \bigcap_{k=r}^{\infty} \mathcal{N}(m_k) = \mathcal{N}(m_r)$. Moreover, $\mathcal{L}^1(m_1) \supseteq \mathcal{L}^1(m_2) \supseteq \ldots$ together with (5) and (10) yields

$$\mathcal{L}^1(m) = \bigcap_{k=1}^{\infty} \mathcal{L}^1(m_k) = \bigcap_{k=r}^{\infty} \mathcal{L}^1(m_k) = \mathcal{L}^1(m_r).$$

Accordingly,

$$L^1(m) = \mathcal{L}^1(m)/\mathcal{N}(m) = \mathcal{L}^1(m_r)/\mathcal{N}(m_r) = L^1(m_r).$$

But, the compactness of I_{m_r}, with m_r being Banach-space-valued, implies that m_r has finite variation and $L^1(m_r)$ is isomorphic to $L^1(|m_r|)$; see Theorems 1 and 4 of [17]. Hence, $L^1(m)$ is isomorphic to the Banach AL-lattice $L^1(|m_r|)$.

Since ω is non-normable and subspaces of a Banach space are normable, it follows that $L^1(m)$ cannot contain an isomorphic copy of the Fréchet space ω. ■

The remainder of this section is devoted to Bochner integrable functions, in so far as we record some appropriate results needed to prove the main theorems.

Let X be a Fréchet space and $\lambda : \Sigma \to [0,\infty)$ be a finite measure defined on a measurable space (Ω, Σ). A function $G : \Omega \to X$ is called *strongly measurable* (for λ) if there are Σ-simple functions $H_n : \Omega \to X$, for $n \in \mathbb{N}$, with $\lim_{n\to\infty} H_n(w) = G(w)$, in X, for λ-a.e. $w \in \Omega$. A strongly measurable function $G : \Omega \to X$ is said to be *Bochner λ-integrable* if $\int_\Omega (p_k \circ G)\, d\lambda < \infty$ for each $k \in \mathbb{N}$, where $p_1 \leq p_2 \leq \ldots$ are continuous seminorms determining the topology of X. Equivalently, there exists a sequence of Σ-simple functions $G_n : \Omega \to X$, for $n \in \mathbb{N}$, which converges pointwise λ-a.e. to G and $\int_\Omega p_k \circ (G - G_n)\, d\lambda \to 0$ as $n \to \infty$, for each $k \in \mathbb{N}$, [18, Lemma 2.5]. For each $E \in \Sigma$, the “integral of G over E” is defined to be the element $\int_E G\, d\lambda := \lim_{n\to\infty} \int_E G_n\, d\lambda$ of X, which exists by the completeness of X and by using the obvious definition of $\int_E G_n\, d\lambda$. This definition is independent of the choice of the sequence $\{G_n\}_{n=1}^\infty$. Furthermore, the *indefinite Bochner λ-integral* $G{\cdot}\lambda : E \mapsto \int_E G\, d\lambda$ is an X-valued vector measure of finite variation and satisfies

$$|(G{\cdot}\lambda)_k|(E) = \int_E (p_k \circ G)\, d\lambda, \qquad E \in \Sigma, \tag{11}$$

for each $k \in \mathbb{N}$, [18, Lemma 2.8].

A function $G : \Omega \to X$ is called *Pettis λ-integrable* if it is *weakly measurable* (i.e., the scalar function $\langle G, x'\rangle : w \mapsto \langle G(w), x'\rangle$ is Σ-measurable for all $x' \in X'$), if $\langle G, x'\rangle \in L^1(\lambda)$ for each $x' \in X'$, and if for each $E \in \Sigma$ there exists a vector (P)-$\int_E G\, d\lambda$ in X satisfying

$$\left\langle (P)\text{-}\int_E G\, d\lambda, x' \right\rangle = \int_E \langle G, x'\rangle\, d\lambda, \qquad x' \in X',$$

[2, p. 88]. By the Orlicz-Pettis theorem, the *indefinite Pettis λ-integral* $E \mapsto (P)$-$\int_E G\, d\lambda$ is an X-valued vector measure on Σ. Every Bochner λ-integrable function is clearly Pettis λ-integrable, but not conversely in general.

A vector measure $m : \Sigma \to X$ of finite variation is called *λ-continuous* if, for each $k \in \mathbb{N}$, we have $|m_k|(E) \to 0$ whenever $\lambda(E) \to 0$, [3, p. 336]. In particular, m is always ν_m-continuous for the finite measure $\nu_m : \Sigma \to [0,\infty)$ defined by

$$\nu_m(E) := \sum_{k=1}^{\infty} |m_k|(E)/2^k\big(1 + |m_k|(\Omega)\big), \qquad E \in \Sigma. \tag{12}$$

The following folklore result presents a different characterization of nuclear Fréchet spaces than that given in Theorem 1.

Proposition 3 *Let X be a Fréchet space. Then X is nuclear if and only if, for each finite, positive measure space $(\Omega, \Sigma, \lambda)$ the class of all X-valued Pettis λ-integrable functions on Ω coincides with the class of all X-valued Bochner λ-integrable functions on Ω.*

PROOF. Let X be nuclear and let $(\Omega, \Sigma, \lambda)$ be any finite, positive measure space. Then every X-valued Pettis λ-integrable function on Ω is known to be Bochner λ-integrable, [21, Theorem 6]. As already noted, Bochner λ-integrable functions are always Pettis λ-integrable and so these two classes of λ-integrable functions coincide.

Suppose now that for each finite, positive measure space $(\Omega, \Sigma, \lambda)$, the classes of X-valued Pettis λ-integrable and Bochner λ-integrable functions coincide. Let $\{x_n\}_{n=1}^\infty$ be any unconditionally summable sequence in X. Let $\Omega := \mathbb{N}$ and $\Sigma := 2^{\mathbb{N}}$, and define a finite, positive measure λ on Σ by $\lambda(E) :=$

$\sum_{n\in E} n^{-2}$, for each $E \in \Sigma$. The function $G : \Omega \to X$ defined by $G(n) := n^2 x_n$ for $n \in \Omega$ is clearly weakly measurable, since

$$\langle G, x' \rangle : n \mapsto \langle G(n), x' \rangle = n^2 \langle x_n, x' \rangle, \qquad n \in \Omega,$$

for each $x' \in X'$. Moreover, $\langle G, x' \rangle \in L^1(\lambda)$ for each $x' \in X'$, because

$$\int_\Omega |\langle G, x' \rangle| \, d\lambda = \sum_{n=1}^{\infty} |\langle G(n), x' \rangle| \, \lambda(\{n\}) = \sum_{n=1}^{\infty} |\langle x_n, x' \rangle|$$

is finite; this follows from the unconditional summability of $\{x_n\}_{n=1}^\infty$ and Riemann's theorem, stating that a sequence of complex numbers is absolutely summable if and only if it is unconditionally summable. Finally, if we define (P)-$\int_E G \, d\lambda := \sum_{n\in E} x_n$, for each $E \in \Sigma$, then it is routine to verify the identity $\langle (P)\text{-}\int_E G \, d\lambda, x' \rangle = \int_E \langle G, x' \rangle \, d\lambda$ for each $x' \in X'$. Accordingly, G is Pettis λ-integrable and hence, by hypothesis, also Bochner λ-integrable. In particular, the indefinite Bochner λ-integral $G{\cdot}\lambda$ has finite variation and, for $k \in \mathbb{N}$ arbitrary, (11) implies that

$$\sum_{n=1}^{\infty} p_k(x_n) = \int_\Omega (p_k \circ G) \, d\lambda < \infty.$$

So, $\{x_n\}_{n=1}^\infty$ is absolutely summable in X. In view of [19, Theorem 4.2.5] it follows that X is nuclear. ■

Proposition 4 *Let X be a Fréchet space and $(\Omega, \Sigma, \lambda)$ be a finite, positive measure space. Let $G : \Omega \to X$ be a Bochner λ-integrable function and m denote the vector measure $G{\cdot}\lambda$. Then*

$$\mathcal{L}^1(|m|) = \{f : \Omega \to \mathbb{C} \textit{ is } \Sigma\textit{-measurable and } fG \textit{ is Bochner-integrable}\} \tag{13}$$

and

$$\mathcal{L}^1(m) = \{f : \Omega \to \mathbb{C} \textit{ is } \Sigma\textit{-measurable and } fG \textit{ is Pettis-integrable}\}. \tag{14}$$

PROOF. To establish (13) let $f : \Omega \to \mathbb{C}$ be a Σ-measurable function such that the function $fG : \Omega \to X$ is Bochner λ-integrable. Clearly fG is strongly measurable. By [18, Lemma 2.8] applied to $m_{fG} := (fG){\cdot}\lambda$ it follows that the vector measure $m_{fG} : \Sigma \to X$ has finite variation and, for each $k \in \mathbb{N}$,

$$|(m_{fG})_k|(E) = \int_E \big(p_k \circ (fG)\big) d\lambda, \qquad E \in \Sigma.$$

That is, $p_k \circ (fG) \in L^1(\lambda)$, for each $k \in \mathbb{N}$. Again by [18, Lemma 2.8], now applied to m, we have

$$\int_\Omega |f| \, d|m_k| = \int_\Omega |f| \, d|(G{\cdot}\lambda)_k| = \int_\Omega |f|{\cdot}(p_k \circ G) \, d\lambda = \int_\Omega \big(p_k \circ (fG)\big) d\lambda < \infty$$

and hence, $f \in \mathcal{L}^1(|m_k|)|$. Since $k \in \mathbb{N}$ is arbitrary it follows that $f \in \bigcap_{k=1}^\infty \mathcal{L}^1(|m_k|) = \mathcal{L}^1(|m|)$.

Conversely, suppose that $f \in \mathcal{L}^1(|m|)$, that is, $f \in \mathcal{L}^1(|m_k|)$ for all $k \in \mathbb{N}$. As already noted, fG is strongly measurable. Moreover, for $k \in \mathbb{N}$, Lemma 2.8 of [18] again applies to yield

$$\int_\Omega \big(p_k \circ (fG)\big) \, d\lambda = \int_\Omega |f|{\cdot}(p_k \circ G) d\lambda = \int_\Omega |f| \, d|m_k| < \infty,$$

which means precisely that fG is Bochner λ-integrable. The identity (13) is thereby established.

The equality (14) is a direct consequence of the various definitions involved combined with the formulae

$$\langle hG, x' \rangle = h \langle G, x' \rangle, \qquad x' \in X',$$

and

$$\langle m, x' \rangle(E) = \int_E \langle G, x' \rangle \, d\lambda, \qquad x' \in X,$$

valid for each $E \in \Sigma$ and each Σ-measurable function $h : \Omega \to \mathbb{C}$. ■

For Banach space versions of the previous result we refer to Propositions 8 and 13 of [20].

3. Proofs of main theorems

We begin immediately with the proof of Theorem 1.

Suppose first that X is a Fréchet space with the property that $L^1(m)$ is a Fréchet AL-lattice for every X-valued vector measure m. According to Lemma 3, $L^1(m)$ is then isomorphic to $L^1(|m|)$. Since the function $\mathbf{1}$ (constantly equal to 1 on Ω) belongs to $L^1(m)$ it also belongs to $L^1(|m|)$ and hence, $\mathbf{1} \in L^1(|m_k|)$ for every $k \in \mathbb{N}$. So, m has finite variation. Then Corollary 4.3 of [15], together with the paragraph immediately following its proof, imply that X is nuclear.

For the proof of the converse implication, suppose that X is a nuclear Fréchet space (with topology determined by the seminorms $p_1 \leq p_2 \leq \ldots$) and that $m : \Sigma \to X$ is a vector measure. Again by [15, Corollary 4.3] it follows that m has finite variation. Let $\nu_m : \Sigma \to [0, \infty)$ be the finite, positive measure given by (12), in which case m is ν_m-continuous; see Section 2. Since nuclear Fréchet spaces have the *Radon-Nikodým property*, [3, p. 338], there exists a function $G : \Omega \to X$ which is integrable by seminorm (relative to ν_m), in the sense of [3, p. 336], and satisfies

$$m(E) = \int_E G \, d\nu_m, \qquad E \in \Sigma. \tag{15}$$

Nuclear Fréchet spaces are necessarily separable, [19, p. 82], and hence, are Suslin spaces. So Proposition 2.3 of [2] implies that G is strongly measurable. Moreover, $\int_\Omega (p_k \circ G) \, d\nu_m < \infty$ for each $k \in \mathbb{N}$, [2, Proposition 2.5(ii)]. Accordingly, G is Bochner ν_m-integrable. Moreover, $m = G{\cdot}\nu_m$ by (15).

Now let $f \in \mathcal{L}^1(m)$, in which case f is Σ-measurable and $fG : \Omega \to X$ is Pettis ν_m-integrable; see (14) of Proposition 4. Then Proposition 3 implies that fG is also Bochner ν_m-integrable. By (13) of Proposition 4 we conclude that $f \in \mathcal{L}^1(|m|)$. This shows that $\mathcal{L}^1(m) \subseteq \mathcal{L}^1(|m|)$ and hence, that $L^1(m) \subseteq L^1(|m|)$. Since the reverse inclusion $L^1(|m|) \subseteq L^1(m)$ always holds, we conclude that $L^1(m) = L^1(|m|)$ as ordered vector spaces. But, the identity map from the Fréchet space $L^1(|m|)$ onto the Fréchet space $L^1(m)$ is continuous and so the open mapping theorem ensures that $L^1(m)$ and $L^1(|m|)$ are isomorphic. According to Lemma 3 we conclude that $L^1(m)$ is a Fréchet AL-lattice. This concludes the proof of Theorem 1. ■

For the case when X is nuclear Fréchet , the inclusion $\mathcal{L}^1(m) \subseteq \mathcal{L}^1(|m|)$ also follows from part (3) of Corollary 4.3 in [15]. In the previous proof we provided an alternative argument of this fact, since our method can also be applied in other settings.

We now turn to the proof of Theorem 2. So, suppose that X is a Fréchet Montel space and m is an X-valued vector measure.

(i) $\Longleftrightarrow$ (ii) is true in every Fréchet space; see Corollary 1.

(ii) $\Longrightarrow$ (iii) is obvious.

(iv) $\Longrightarrow$ (i) is true in every Fréchet space; see Proposition 2.

(iii) $\Longrightarrow$ (iv). Since the integration operator I_m is always continuous, the normability of $L^1(m)$ implies that the open unit ball B of the norm determining the topology of $L^1(m)$, with B also a *bounded* subset of $L^1(m)$, gets mapped by I_m to a bounded subset of X. Since X is Montel, $I_m(B)$ is relatively compact in X and hence, I_m is a compact operator. The proof of Theorem 2 is thereby complete. ■

Remark 1 In statement (i) of Theorem 2 it is *not* possible to replace "Banach AL-lattice" with "Fréchet AL-lattice". Indeed, let $X := \omega = \mathbb{C}^{\mathbb{N}}$ be equipped with the topology determined by the increasing sequence of seminorms

$$p_k(x) := \sum_{j=1}^{k} |x_j|, \qquad x = (x_1, x_2, \ldots) \in X,$$

for each $k \in \mathbb{N}$, in which case X is a nuclear (hence, Montel) Fréchet space. Let $\Omega := \mathbb{N}$ and $\Sigma := 2^{\mathbb{N}}$, and define an X-valued vector measure $m : \Sigma \to X$ by

$$m(E) := \big(\chi_E(1), \chi_E(2), \ldots\big), \qquad E \in \Sigma.$$

Observe that $\mathcal{N}(m) = \{0\}$ and so $\mathcal{L}^1(m) = L^1(m)$. Moreover, $L^1(m) = \mathbb{C}^{\mathbb{N}}$ as an ordered vector space and, for any function $f : \mathbb{N} \to \mathbb{C}$, we have $\int_E f\, dm = f\chi_E$ for $E \in \Sigma$. Hence, for $f \in L^1(m)$ and $k \in \mathbb{N}$ it is the case that

$$p_k(m)(f) = \sup\left\{p_k(f\chi_E) : E \in \Sigma\right\} = \sum_{j=1}^{k} |f(j)| = p_k(f).$$

This shows that $L^1(m)$ is isomorphic to X. Since the lattice seminorms $p_1, p_2, \ldots$ have the AL-property, we see that $L^1(m)$ is a Fréchet AL-lattice. However, $L^1(m)$ is not normable and so (iii) of Theorem 2 fails. ■

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S. Okada
Department of Mathematics
Macquarie University
Sydney, NSW, 2109
Australia
okada@ics.mq.edu.au

W. J. Ricker
Mathem.-Geogr. Fakultät
Katholische Universität Eichstätt-Ingolstadt
D – 85071 Eichstätt
Germany
werner.ricker@ku-eichstaett.de