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On suprabarrelledness of $c_0(\Omega, X)$

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Abstract. Assuming that Ω is a non-empty set and X is a real or complex normed space, we show that the linear space $c_0(\Omega, X)$ of all functions $f : \Omega \to X$ such that for each $\varepsilon > 0$ the set $\{\omega \in \Omega : \|f(\omega)\| > \varepsilon\}$ is finite, endowed with the supremum norm, is suprabarrelled if and only if X is suprabarrelled.

Supratonelación en $c_0(\Omega, X)$

Resumen. Si Ω es un conjunto no vacío y X es un espacio normado real o complejo, se tiene que, con la norma supremo, el espacio $c_0(\Omega, X)$ formado por las funciones $f : \Omega \to X$ tales que para cada $\varepsilon > 0$ el conjunto $\{\omega \in \Omega : \|f(\omega)\| > \varepsilon\}$ es finito es supratonelado si y sólo si X es supratonelado.

1. Preliminaries

Along this paper Ω will denote a non-empty set and X a normed space over the field $\mathbb{K}$ of real or complex numbers. We represent by $c_0(\Omega, X)$ the linear space over $\mathbb{K}$ of all those functions $f : \Omega \longrightarrow X$ such that for each $\varepsilon > 0$ the set $\{\omega \in \Omega : \|f(\omega)\| > \varepsilon\}$ is finite or empty, equipped with the supremum norm $\|f\|_\infty = \sup\{\|f(\omega)\| : \omega \in \Omega\}$. Since the support of f is $\bigcup_{n=1}^{\infty} \left\{\omega \in \Omega : \|f(\omega)\| > \frac{1}{n}\right\}$, each $f \in c_0(\Omega, X)$ is countably supported.

If Γ is a (possibly empty) subset of Ω, we denote by $c_0(\Gamma, X)$ the linear subspace of $c_0(\Omega, X)$ consisting of all those functions f with $f(\Omega - \Gamma) = \{0\}$.

If Ω is countable infinite, then we shall write $c_0(X)$ instead of $c_0(\Omega, X)$ ([10]). Hence $c_0(X)$ is the linear space of all sequences in X convergent to zero, endowed with the supremum norm.

Let us recall that a (Hausdorff) locally convex space E is barrelled if each barrel (*i.e.* each absorbing closed absolutely convex set) in E is a neighbourhood of the origin (see [11], 3.1.2).

An increasing p-web in a set Y (see [1]) is a family $\mathcal{W} = \{E_t : t \in T_p\}$, with $T_p = \bigcup_{k=1}^{p} \mathbb{N}^k$, such that $Y = \bigcup_{n\in\mathbb{N}} E_n$, $E_n \subset E_{n+1}$, $E_t = \bigcup_{n\in\mathbb{N}} E_{t,n}$ and $E_{t,n} \subset E_{t,n+1}$, for $t \in T_{p-1}$ and $n \in \mathbb{N}$. If Y is a vector space and every E_t is a linear subspace of Y, we say then that $\mathcal{W}$ is a linear increasing p-web.

A (Hausdorff) locally convex space E is called p-barrelled if given in E a linear increasing p-web $\mathcal{W} = \{E_t : t \in T_p\}$ there is a $t \in \mathbb{N}^p$ such that E_t is barrelled and dense in E (see [4], [6], [8], [12] and [13]). The 1-barrelled spaces were introduced by Valdivia ([18]) with the name of suprabarrelled spaces, called (db) in [14] and [16].

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In [7] it is proved that $c_0(\Omega, X)$ is either barrelled, ultrabornological, or unordered Baire-like (UBL for short, [17]) if and only if X is, respectively, barrelled, ultrabornological or UBL. The case of real or complex continuous functions spaces defined on a locally compact space and vanishing at infinity is considered in [3]. Before [7] there were only a few examples of vector valued functions spaces which were UBL whenever X is (non-complete) UBL. It is natural to ask whether or not the preceding considerations are also true in the class of suprabarrelled spaces, because UBL spaces are suprabarrelled ([6], 3.1 and 3.2.2) and suprabarrelled spaces enjoy useful properties, for instance it is well known that the linear mappings with closed graph from a suprabarrelled space into a (LB)-space have strong localizations properties (see, for instance, [5], [9], [15], [17], [18] and [20]). In this paper we shall prove that $c_0(\Omega, X)$ is suprabarrelled if and only if X is suprabarrelled. Also a new property about linear increasing $1-$webs in $c_0(\Omega, X)$ will be obtained. In what follows $\langle V \rangle$ means the linear hull of V. We are going to use the classical notation given, for instance, in [2] and [19].

2. Suprabarrelledness in $c_0(\Omega, X)$

Let us suppose that $c_0(\Omega, X)$ is the union of an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of subspaces. Let T_n be a barrel in F_n, $V_n = \overline{T_n}^{c_0(\Omega,X)}$, $Z_n = \langle V_n \rangle$ and $S_n = \bigcap \{Z_m : m \geq n\}$. From $F_n \subset S_n$ it follows that $c_0(\Omega, X) = \bigcup_{n=1}^{\infty} S_n$.

Lemma 1 *If F is a suprabarrelled subspace of $c_0(\Omega, X)$ there exists $n \in \mathbb{N}$ such that $F \subset S_n$.*

PROOF. $\{F \cap S_n : n \in \mathbb{N}\}$ is an increasing covering of F. The suprabarrelledness implies that there is an $F \cap S_n$ which is barrelled and dense in F. Then, if $m \geq n$ we have that $\overline{T_m}^{c_0(\Omega,X)}$ contains a neighbourhood of 0 in $F \cap S_n$. By density we have that $\overline{T_m}^{c_0(\Omega,X)}$ also contains a neighbourhood of 0 in F. This implies that $F \subset Z_m$ when $m \geq n$. It follows that $F \subset \bigcap \{Z_m : m \geq n\} = S_n$. ■

The preceding lemma has the following obvious extension which will be useful in the end of Proposition 1.

Lemma 2 *Let F be a subspace of $c_0(\Omega, X)$ and $\mathcal{T}$ a locally convex topology in F finer than the induced by $c_0(\Omega, X)$. If $(F, \mathcal{T})$ is suprabarrelled there exists $n \in \mathbb{N}$ such that $F \subset S_n$.*

PROOF. Since $\{F \cap S_n : n \in \mathbb{N}\}$ is an increasing covering of $(F, \mathcal{T})$ there exists an $(F \cap S_n, \mathcal{T}_{F \cap S_n})$ which is barrelled and dense in $(F, \mathcal{T})$. If $n \leq m$, then $F \cap S_n \cap \overline{T_m}^{c_0(\Omega,X)}$ is a barrel in $F \cap S_n$ endowed with the topology induced by $\mathcal{T}$. Therefore $\overline{T_m}^{c_0(\Omega,X)}$ contains a neighbourhood of 0 in $(F, \mathcal{T})$, implying that $F \subset Z_m$ when $m \geq n$. Then $F \subset \bigcap \{Z_m : m \geq n\} = S_n$. ■

Proposition 1 *There exists a finite set Δ (possibly empty) and a natural n such that $c_0(\Omega \diagdown \Delta, X) \subset S_n$.*

PROOF. First step: We are going to prove that there exists a countable set Δ and a natural number n such that $c_0(\Omega \diagdown \Delta, X) \subset S_n$. If this were not true, there would be a $f_1 \in c_0(\Omega, X)$ with $\|f_1\|_\infty = 1$ and $f_1 \notin S_1$.

The set $\Delta_1 = \operatorname{supp}(f_1)$ is countable and from $c_0(\Omega \diagdown \Delta_1, X) \nsubseteq S_2$ we deduce the existence of a $f_2 \in c_0(\Omega \diagdown \Delta_1, X)$ with $\|f_2\|_\infty = 1$ and $f_2 \notin S_2$. The set $\Delta_2 = \operatorname{supp}(f_2)$ is countable and we choose $f_3 \in c_0(\Omega \diagdown \{\Delta_1 \cup \Delta_2\}, X)$ with $\|f_3\|_\infty = 1$ and $f_3 \notin S_3$.

By induction we would obtain a bounded sequence $\{f_n : n \in \mathbb{N}\}$ in $c_0(\Omega, X)$ and a pairwise disjoint sequence $\{\Delta_n : n \in \mathbb{N}\}$ of countable subsets of Ω such that $\Delta_n = \operatorname{supp}(f_n)$, $\|f_n\|_\infty = 1$ and $f_n \notin S_n$ for each $n \in \mathbb{N}$.

The mapping φ from c_0 into $c_0(\Omega, X)$ defined by $\varphi(\{\xi_n : n \in \mathbb{N}\}) = \sum_{n=1}^{\infty} \xi_n f_n$ is well-defined since $\{\xi_n : n \in \mathbb{N}\} \in c_0$, $\{f_n : n \in \mathbb{N}\}$ is bounded and for $\omega \in \Omega$ we have that $\sum_{n=1}^{\infty} \xi_n f_n(\omega)$ has at most one non-null term. It is also obvious that φ is an isometry onto and then, by Lemma 1, we have that there exists an $n \in \mathbb{N}$ such that $\varphi(c_0) \subset S_n$. Then the relation $f_n \in \varphi(c_0) \subset S_n$ contradicts the choice $f_n \notin S_n$ and with this contradiction concludes the first part of the proof. Without loss of generality we may suppose that $\Delta = \mathbb{N}$.

Second step: We are going to prove that there is a natural number i such that $c_0(\mathbb{N} \diagdown \{1, 2, 3, \ldots, i\}, X) \subset S_i$.

In fact, if $c_0(\mathbb{N} \diagdown \{1, 2, 3, \ldots, i\}, X) \nsubseteq S_i$, for $i = 1, 2, 3, ...$ then there exists a sequence $\{f_i : i \in \mathbb{N}\}$ such that $f_i \in c_0(\mathbb{N} \diagdown \{1, 2, 3, ..., i\}, X) - S_i$ and $\|f_i\|_\infty = 1$ for $i = 1, 2, 3, \ldots$. Then the mapping φ from l_1 into $c_0(\mathbb{N}, X)$ defined by

$$\varphi(\{\xi_n : n \in \mathbb{N}\}) = \sum_{n=1}^{\infty} \xi_n f_n$$

is well-defined since $\sum_{n=1}^{\infty} \xi_n f_n(i)$ has at most $i - 1$ non-null terms, and $\sum_{n=1}^{\infty} \xi_n f_n \in c_0(\mathbb{N}, X)$ because if $\varepsilon > 0$ there is a k such that $\sum_{n=k}^{\infty} |\xi_n| < \frac{\varepsilon}{2}$, implying that $\|\sum_{n=k}^{\infty} \xi_n f_n\|_\infty < \frac{\varepsilon}{2}$. Since $\sum_{n=1}^{k-1} \xi_n f_n \in c_0(\mathbb{N}, X)$ there exists a $p \in \mathbb{N}$ such that $\left\|\sum_{n=1}^{k-1} \xi_n f_n(i)\right\| < \frac{\varepsilon}{2}$ when $i \geq p$. Therefore $\|\sum_{n=1}^{\infty} \xi_n f_n(i)\| \leq \left\|\sum_{n=1}^{k-1} \xi_n f_n(i)\right\| + \|\sum_{n=k}^{\infty} \xi_n f_n(i)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $i \geq p$, which proves that $\sum_{n=1}^{\infty} \xi_n f_n \in c_0(\mathbb{N}, X)$.

If $\xi \in l_1$ we have that $\|\varphi(\xi)\| \leq \|\xi\|_1$. The continuity of φ enables us to consider in $\varphi(l_1)$ the finest locally convex topology $\mathcal{T}$ such that $\varphi : l_1 \to (\varphi(l_1), \mathcal{T})$ is continuous. Then $\mathcal{T}$ is finer than the topology induced by $c_0(\mathbb{N}, X)$ and $(\varphi(l_1), \mathcal{T})$ is suprabarrelled because it is isometric to a quotient of l_1. Then, by Lemma 2, there exists an $n \in \mathbb{N}$ such that $\varphi(l_1) \subset S_n$, giving the contradiction $f_n \in \varphi(l_1) \subset S_n$. This establishes this second step.

The two steps give directly the proposition. ■

Theorem 1 *If X is suprabarrelled there is a natural n such that $c_0(\Omega, X) = S_n$.*

PROOF. By the preceding proposition we only need to prove that if Δ is a finite set there is a $n \in \mathbb{N}$ such that $c_0(\Delta, X) \subset S_n$. This follows from the Lemma 1, the isomorphism $c_0(\Delta, X) = X^\Delta$ and the fact that the product of suprabarrelled spaces is suprabarrelled (see Proposition 3.2.10 in [6]) ■

Theorem 2 *X is suprabarrelled if and only if $c_0(\Omega, X)$ is suprabarrelled, being Ω a non-void set.*

PROOF. For $p \in \Omega$, the spaces X and the quotient $c_0(\Omega, X) \diagup c_0(\Omega \diagdown \{p\}, X)$ are isometric. Then, if $c_0(\Omega, X)$ is suprabarrelled we have that X is suprabarrelled by Proposition 3.2.12 in [6].

Conversely, if X is suprabarrelled we have that $c_0(\Omega, X)$ is Baire-like [15] by Proposition 2.2 in [7] and Proposition 1.2.1 in [6]. Then, if $\{F_n : n \in \mathbb{N}\}$ is a linear increasing $1-$web of $c_0(\Omega, X)$ there is an $p \in \mathbb{N}$ such that F_m is dense in $c_0(\Omega, X)$ for $m \geq p$.

Therefore, if $c_0(\Omega, X)$ were not suprabarrelled we could find an increasing covering $\{F_n : n \in \mathbb{N}\}$ of $c_0(\Omega, X)$, such that each F_n is non-barrelled and dense in $c_0(\Omega, X)$. Let T_n be a barrel in F_n which is not neighbourhood of 0 in F_n. If $V_n = \overline{T_n}^{c_0(\Omega, X)}$ and $S_n = \bigcap_{m \geq n} \langle V_m \rangle$ we have by Theorem 1 that there is an n such that $c_0(\Omega, X) = S_n$.

Then $S_n = \langle V_n \rangle$ and, by Proposition 2.2 in [7], we have that V_n is a neighbourhood of 0 in S_n, implying that T_n is a neighbourhood of 0 in F_n. This contradiction proves the theorem. ■

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