Rev.R.Acad.Cienc.Exact.Fís.Nat. (Esp)
Vol. 93, N.º 2, pp 227-229, 1999
Matemáticas

A NONSTANDARD APPROACH TO ARITHMETIC

(undecidability, incompleteness, inseparability)

JOSÉ F. PRIDA
Departamento de Lógica y Filosofía de la Ciencia, Facultad de Filosofía, Edificio B, Universidad Complutense, 28040 Madrid, Spain

Presentado por J. M. Montesinos, el 22 de mayo de 1996. Aceptado el 24 de marzo de 1999

ABSTRACT

The property of recursive functions of being definable in elemental arithmetic is the usual starting point to study the pathologies of consistent extensions of this theory. However a not so nice property of the much more extensive class of partial recursive functions permits to give simpler proofs of some classical theorems, avoiding in a natural way undesirable technicalities like ω-consistency.

Definition 1. The class of partial recursive functions is the smallest set of funtions containing sum, product, projection functions and the characteristic function of equality and being closed under composition and minimization of a total function.

It is well known that partial recursive functions are those numerical functions which can be computed by programs like Turing machines, Markov's algorithms, etc. Correlating programs with natural numbers, the function from ω to ω defined by the program of number m will be denoted ϕ_m . W_m will be the domain of ϕ_m and $W_{m,s}$, the subset of W_m consisting of those $a \in \omega$ such that the program of number m produces an output after s steps when it starts with input a.

Definition 2. **Q** will denote Robinson Arithmetic, i.e., the theory with the specific set of symbols $\{0, s, +, .\}$, and with the seven usual axioms for the successor function, the sum and the product.

Definition 3. For any $n \in \omega$, $\mathbf{n}$ is the term $ss...s0$ (n times s).

Definition 4. The arithmetical formula $\alpha(x_0, x_1, ..., x_n)$ extends the partial function $\xi : \omega^n \to \omega$ in **Q** iff we have for every $a_1, a_2, ..., a_n, b \in \omega$:

if $\xi(a_1, ..., a_n) = b$, then $\vdash^{Q} \forall x_0(\alpha(x_0, \mathbf{a_1}, ..., \mathbf{a_n}) \leftrightarrow x_0 = \mathbf{b})$.

Extension theorem. Every partial recursive function is extensible in **Q**.

The theorem is an immediate consequence of the following three lemmata whose proofs can be found in [1]:

Lemma 1. The projection functions, the sum, the product, and the characteristic function of the identity are extensible in **Q**.

Let $a = a_1, a_2, ..., a_n$, and $x = x_1, x_2, ..., x_n$.

Lemma 2. If $\psi(a) = \psi_0 (\psi_1(a), ... , \psi_m(a))$, and $\alpha_0, \alpha_1, ..., \alpha_m$ respectively extend $\psi_0, \psi_1, ..., \psi_m \in$ **Q**, then the formula

$\exists y_1 ... \exists y_m (\alpha_1(y_1, x) \wedge ... \wedge \alpha_m(y_m, x) \wedge \alpha_0(x_0, y_1, ..., y_m))$ extends ψ in **Q**.

Lemma 3. If f is a recursive function, $\xi(a_1, ..., a_n) = \mu b[f(a_1, ..., a_n, b) = 0]$ (where μb means *the least b such that*), and $\alpha(x_0, x_1, ..., x_{n+1})$ extends f in **Q**, then the formula

$$\alpha(\mathbf{0}, x_1, ..., x_m, x_0) \wedge \forall y (y < x_0 \to \neg \alpha(\mathbf{0}, x_1, ..., x_m, y))$$

extends ξ in **Q**, where we define $x < y$ to be the formula $\exists z (sz + x = y)$.

Remark. The main difference between the extension theorem for partial recursive functions and the representation theorem for total recursive functions is that in the first case, if **P** is a consistent extension of Q, $\vdash^{P} \forall x_0(\alpha(x_0, \mathbf{a_1}, ..., \mathbf{a_n}) \leftrightarrow x_0 = \mathbf{b})$ implies $\xi(a_1, ..., a_n) = b$ or $\xi(a_1, ..., a_n) \uparrow$, where $\xi(a_1, ..., a_n) \uparrow$ means that $(a_1, ..., a_n) \notin dom(\xi)$.

Let $\psi : \omega \to \omega$ be the partial recursive function such that (first diagonalization!)

$$\psi(n) = \begin{cases} \phi_n(n) & \text{if } n \in dom(\phi_n) \\ \uparrow & \text{otherwise} \end{cases}$$

It follows from the extension theorem:

Corollary 1. There is an arithmetical formula $\alpha(x_0, x_1)$ such that for every consistent extension **P** of **Q** verifies:

If $\psi(a) = b$, then $\vdash^{\mathbf{P}} \forall x_0(\alpha(x_0, \mathbf{a}) \leftrightarrow x_0 = \mathbf{b})$.

Incompleteness theorem (Gödel 1931, Rosser 1936). There is no consistent, axiomatizable and complete extension of **Q**.

Proof. Let **P** be a consistent, axiomatizable extension of **Q**, ψ and $\alpha(x_0, x_1)$ denote the partial recursive function and the formula defined in Corollary 1., $\ulcorner\gamma\urcorner$ denote the Gödel's number of the formula γ, $D = \{\ulcorner\alpha\urcorner : \alpha \in \mathbf{P}\}$ and $R = \{\ulcorner\alpha\urcorner : \neg\,\alpha \in \mathbf{P}\}$. Then let r the natural number such that

$$\phi_r(n) = \begin{cases} 1 \text{ if } \ulcorner\alpha(\mathbf{0}, \mathbf{n})\urcorner \in D \\ 0 \text{ if } \ulcorner\alpha(\mathbf{0}, \mathbf{n})\urcorner \in R \\ \uparrow \text{ otherwise.} \end{cases}$$

Then we have (second diagonalization!),

$$\begin{aligned} \vdash^{\mathbf{P}} \alpha(\mathbf{0}, \mathbf{r}) &\Rightarrow \ulcorner\alpha(\mathbf{0}, \mathbf{r})\urcorner \in D \\ &\Rightarrow \phi_r(r) = 1 \\ &\Rightarrow \psi(r) = 1 \\ &\Rightarrow \vdash^{\mathbf{P}} \forall x_0(\alpha(x_0, \mathbf{r}) \leftrightarrow x_0 = \mathbf{1}) \\ &\Rightarrow \vdash^{\mathbf{P}} \neg\,\alpha(\mathbf{0}, \mathbf{r}). \end{aligned}$$

On the other hand

$$\begin{aligned} \vdash^{\mathbf{P}} \neg\,\alpha(\mathbf{0}, \mathbf{r}) &\Rightarrow \ulcorner\alpha(\mathbf{0}, \mathbf{r})\urcorner \in R \\ &\Rightarrow \phi_r(r) = 0 \\ &\Rightarrow \psi(r) = 0 \\ &\Rightarrow \vdash^{\mathbf{P}} \forall x_0(\alpha(x_0, r) \leftrightarrow x_0 = \mathbf{0}) \\ &\Rightarrow \vdash^{\mathbf{P}} \alpha(\mathbf{0}, \mathbf{r}). \end{aligned}$$

Therefore, it follows from the consistency of **P** that neither $\alpha(\mathbf{0}, \mathbf{r})$ nor $\neg\,\alpha(\mathbf{0}, \mathbf{r})$ are theorems of **P**.

Corollary 2. The theory $Th(\mathcal{N})$ of the structure $\mathcal{N} = (\omega, 0, s, +, .)$ is not axiomatizable and undecidable.

Proof. $Th(\mathcal{N})$ is a consistent and complete extension of **Q**, and decidable implies axiomatizable.

Undecidability theorem (Church, 1936). There is no consistent and decidable extension of **Q**.

Proof. Let **P** be a consistent extension of **Q**, and let D and R be as in the proof of the incompleteness theorem. We prove that there is no recursive set M such that $D \subset M$ and $R \subset \overline{M}$.

Indeed, if such M exists let ψ and $\alpha(x_0, x_1)$ the partial recursive function and the formula defined in Corollary 1., and let s denote the natural number such that

$$\phi_s(n) = \begin{cases} 1 \text{ if } \ulcorner\alpha(\mathbf{0}, \mathbf{n})\urcorner \in M \\ 0 \text{ if } \ulcorner\alpha(\mathbf{0}, \mathbf{n})\urcorner \notin M. \end{cases}$$

Then we have (second diagonalization!),

$$\begin{aligned} \ulcorner\alpha(\mathbf{0}, \mathbf{s})\urcorner \in M &\Rightarrow \phi_s(s) = 1 \\ &\Rightarrow \psi(s) = 1 \\ &\Rightarrow \vdash^{\mathbf{P}} \forall x_0(\alpha(x_0, \mathbf{s}) \leftrightarrow x_0 = \mathbf{1}) \\ &\Rightarrow \vdash^{\mathbf{P}} \neg\,\alpha(\mathbf{0}, \mathbf{s}) \\ &\Rightarrow \ulcorner\alpha(\mathbf{0}, \mathbf{s})\urcorner \in R \subset \overline{M}. \end{aligned}$$

On the other hand

$$\begin{aligned} \ulcorner\alpha(\mathbf{0}, \mathbf{s})\urcorner \notin M &\Rightarrow \phi_s(s) = 0 \\ &\Rightarrow \psi(s) = 0 \\ &\Rightarrow \vdash^{\mathbf{P}} \forall x_0(\alpha(x_0, \mathbf{s}) \leftrightarrow x_0 = \mathbf{0}) \\ &\Rightarrow \vdash^{\mathbf{P}} \alpha(\mathbf{0}, \mathbf{s}) \\ &\Rightarrow \ulcorner\alpha(\mathbf{0}, \mathbf{s})\urcorner \in D \subset M. \end{aligned}$$

Therefore we have:

$\ulcorner\alpha(\mathbf{0}, \mathbf{s})\urcorner \in M$ iff $\ulcorner\alpha(\mathbf{0}, \mathbf{s})\urcorner \notin M$, a contradiction.

Both Rosser's and Church's theorems are an immediate consequence of a much stronger inseparability result.

Definition 5. Two disjoint sets $A, B \subset \omega$ are recursively inseparable if there is no recursive set C such that $A \subset C$ and $B \subset \overline{C}$.

For every pair of disjoint sets A, B, the following propositions are equivalent:

[a] A and B are recursively inseparable.

[b] $\neg\exists x\, \exists y\, (A \subset W_x \,\&\, B \subset W_y \,\&\, W_x \cap W_y = \emptyset \,\&\, W_x \cup W_y = \omega)$

[c] $\forall x\, \forall y\, ((A \subset W_x \,\&\, B \subset W_y \,\&\, W_x \cap W_y = \emptyset) \to \exists z(z \in \overline{W}_x \cap \overline{W}_y))$

[d] $\exists F\, \forall x\, \forall y\, ((A \subset W_x \,\&\, B \subset W_y \,\&\, W_x \cap W_y = \emptyset) \to F(x, y) \in \overline{W}_x \cap \overline{W}_y))$

(axiom of choice!).

The equivalence between [**a**] and [**d**] justifies the following

Definition 6. Two disjoint sets $A, B \subset \omega$ are effectively inseparable if there is a recursive function f verifying

$$\forall x\ \forall y\ ((A \subset W_x\ \&\ B \subset W_y\ \&\ W_x \cap W_y = \emptyset) \to f(x, y) \in \overline{W_x} \cap \overline{W_y}).$$

Inseparability theorem. For every consistent and axiomatizable extension **P** of **Q** the sets $D = \{\ulcorner\alpha\urcorner : \alpha \in \mathbf{P}\}$ and $R = \{\ulcorner\alpha\urcorner : \neg\alpha \in \mathbf{P}\}$ are effectively inseparable.

Proof. Let ψ and $\alpha(x_0, x_1)$ be the partial recursive function and the formula defined in Corollary 1 and let f be the recursive function defined by the identity $f(a) = \ulcorner\alpha(\mathbf{0}, \mathbf{a})\urcorner$. By the recursion theorem there is a recursive function h such that

$$\phi_{h(a,b)}(n) = \begin{cases} 0 & \text{if } \exists s\big(fh(a,b) \in W_{b,s} - W_{a,s}\big) \\ 1 & \text{if } \exists s\big(fh(a,b) \in W_{a,s} - W_{b,s}\big) \\ \uparrow & \text{otherwise.} \end{cases}$$

It is clear that fh is the inseparability function of D and R, because if

$$D \subset W_a,\ R \subset W_b \text{ and } W_a \cap W_b = \emptyset$$

then

$$fh(a, b)\ \overline{W_a} \cap \overline{W_b}.$$

Indeed

$$\begin{aligned} fh(a, b) \in W_a &\Rightarrow \phi_{h(a,b)}(h(a, b)) = 1 \\ &\Rightarrow \psi(h(a, b)) = 1 \\ &\Rightarrow \vdash^{\mathbf{P}} \forall x_0(\alpha(x_0, \mathbf{h(a, b)}) \leftrightarrow x_0 = \mathbf{1}) \\ &\Rightarrow \vdash^{\mathbf{P}} \neg\, \alpha(\mathbf{0}, \mathbf{h(a, b)}) \\ &\Rightarrow \ulcorner\alpha(\mathbf{0}, \mathbf{h(a, b)})\urcorner = fh(a, b) \in R \subset W_b \subset \overline{W_a}, \end{aligned}$$

a contradiction.

Similarly

$$\begin{aligned} fh(a, b) \in W_b &\Rightarrow \phi_{h(a,b)}(h(a, b)) = 0 \\ &\Rightarrow \psi(h(a, b)) = 0 \\ &\Rightarrow \vdash^{\mathbf{P}} \forall x_0(\alpha(x_0, \mathbf{h(a, b)}) \leftrightarrow x_0 = \mathbf{0}) \\ &\Rightarrow \vdash^{\mathbf{P}} \alpha(\mathbf{0}, \mathbf{h(a, b)}) \\ &\Rightarrow \ulcorner\alpha(\mathbf{0}, \mathbf{h(a, b)})\urcorner = fh(a, b) \in D \subset W_a \subset \overline{W_b}, \end{aligned}$$

a contradiction.

Corollary 3 (Church). No consistent extension of **Q** is decidable. The proof is obvious.

Corollary 4 (Rosser). No consistent, axiomatizable extension **P** of **Q** is complete.

Proof. Let D, R, f, h be as in the proof of the inseparability theorem. If L is the set of all arithmetical closed formulae, and we define G, a, and b by the identities

$$G = \{\ulcorner\alpha\urcorner : \alpha \in L\},$$

$$W_a = D,$$

$$W_b = R \cup \overline{G},$$

it is obvious that

$$D \subset W_a,\ R \subset W_b \text{ and } W_a \cap W_b = \emptyset$$

and hence

$$fh(a, b) \in \overline{W_a} \cap \overline{W_b} = G \cap \overline{D} \cap \overline{R}.$$

$fh(a, b) \in G$ means that there exists a formula α such that $\ulcorner\alpha\urcorner = fh(a, b)$. And it follows from $fh(a, b) \in \overline{D} \cap \overline{R}$ that $\alpha \notin \mathbf{P}$ and $\neg\alpha \notin \mathbf{P}$.

REFERENCES

1. Boolos, George S. & Jeffrey, Richard C. (1989) *Computability and Logic,* Third edition, Cambridge University Press, Cambridge.

SERIE «LIBROS ANTIGUOS»
REAL ACADEMIA DE CIENCIAS

Quintana, Jerónimo de

A la muy antigua, noble y coronada villa de Madrid : historia de su antiguedad, nobleza y grandeza / por el licenciado Geronimo de Quintana... - Con privilegio en Madrid : en la Imprenta del Reyno, 1629.