(Φ, φ)-ABSOLUTELY SUMMING OPERATORS

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Se introduce una clase de operadores que generalizan las clases de operadores φ-absolutamente sumables, Φ-absolutamente sumables y se presentan unas propiedades.

In this paper a new class of absolutely summing operators is introduced wich is more general than the classes wich are introduced in the papers [1], [2], [3], [6].

Let E, F be normed spaces and $T : E \longrightarrow F$ a *linear and bounded* operator ($T \in L(E, F)$).

The functions (*) Φ of R. Schatten are defined in [5], [4] and the functions φ in the paper [3].

Let R be the field of real numbers and c_0 the space of all zero sequences

$$(x = \{ x_i \} \in c_0 \quad \text{if} \quad \lim x_i = 0).$$

$\hat{c}$ is the subspace of c_0 wich contains the sequences of finite rank

$$(\{ x_i \} \in \hat{c} \quad \text{if} \quad x = \{ x_1, \ldots, x_{n}, 0, 0, \ldots \}, \ n < \infty).$$

The properties of the function Φ are the following

$$\Phi : \hat{c} \longrightarrow R_+ ; \quad \Phi(x + y) \leq \Phi(x) + \Phi(y), \quad x, y \in \hat{c};$$
$$\Phi(\lambda x) = |\lambda| \Phi(x), \quad \lambda \in R, \quad x \in \hat{c};$$
$$\Phi(1, 0, 0, \ldots) = 1; \quad \Phi(x_1, x_2, \ldots, x_n, 0, 0, \ldots) =$$
$$= \Phi(|x_{i_1}|, |x_{i_2}|, \ldots, |x_{i_n}|, 0, 0, \ldots),$$

(*) Norm (norming) functions.

where

$$i_1, i_2, \ldots, i_n$$

is a permutation of

$$1, 2, \ldots, n \cdot ; \Phi(x) = 0 \quad \text{iff} \quad x = 0,$$

The functions φ posses the following properties

$\varphi : R_+ \longrightarrow R_+$; $\varphi(x+y) \leq \varphi(x) + \varphi(y)$; $\varphi(x) < \varphi(y)$ iff $x < y$; $\varphi(0) = 0$; φ is continuous.

The conjugate function of Φ with respect to the function Ψ is

$$\Phi^*_\psi(x) = \sup_{y \in \hat{k}} \frac{\Psi(x\,y)}{\Phi(y)}, \quad \hat{k} = \{ x \in \hat{c} \mid x_i \geq 0 \}, \quad x\,y = \{ x_1 y_1, x_2 y_2, \ldots \} \text{ [4]}$$

$$\left(\text{In the particular case } \Phi^*(x) = \sup_{y \in \hat{k}} \frac{\Sigma x_i y_i}{\Phi(y)} \text{ [5]} \right).$$

DEFINITION 1.1.—The operator $T \in L(E, F)$ is called (Φ, φ) — absolutely summing if for all sequence of finite rank $\{x_i\} \in E$ exists the constant $c \geq 0$ such that

$$\Phi(\varphi(\|T x_i\|)) \leq c \sup_{\|a\| \leq 1} \Phi(\varphi(|\langle x_i, a \rangle|)), \quad a \in E', \quad \text{where } E'$$

is the conjugate space of E.

This class is denoted $\pi_{\Phi,\varphi}(E, P)$.

REMARK.—If

$$\Phi(x) = \Phi_1(x) = \Sigma |x_i|$$

results the class of φ — absolutely summing operators [3] and if $\varphi(x) = x$ results the class of absolutely Φ — summing operators [6] (see also the Λ — summing operators). More particular if

$$\Phi(x) = \Phi_p(x) = (\Sigma |x_i|^p)^{\frac{1}{p}}, \quad p \geq 1, \quad \varphi(t) = t$$

results the class of p — absolutely summing operators [2].

PROPOSITION 1.1.—$\pi_{\Phi,\varphi}$ (E, F) is a quasinormed space with the quasinorm

$$\pi_{\phi,\varphi}(T) = \inf \{ c \geq 0 \mid \Phi(\varphi(\|T x_i\|)) \leq c \sup_{\|a\| \leq 1} \Phi(\varphi(|\langle x_i, a \rangle|)) \}.$$

PROOF.—Let be

$$T_k \in \pi_{\Phi,\varphi} \quad (k = 1, 2)$$

Hence

$$\Phi(\varphi(\| T_k x_i \|)) \leq c_k \quad \sup \quad \Phi(\varphi(|\langle x_i, a \rangle|)), \quad (k = 1, 2)$$

and

$$\Phi(\varphi(\|(T_1 + T_2) x_i\|)) \leq \Phi(\varphi(\|T_1 x_i\|)) + \Phi(\varphi(\|T_2 x_i\|)) \leq$$
$$\leq (c_1 + c_2) \sup_{\|a\| \leq 1} \Phi(\varphi(|\langle x_i, a \rangle|)$$

Thus

$$\pi_{\Phi,\varphi}(T_1 + T_2) \leq \pi_{\phi,\varphi}(T_1) + \pi_{\Phi,\varphi}(T_2).$$

Also if $\lambda \in R$ and

$$T \in \pi_{\phi,\varphi}(E, F)$$

results

$$\Phi(\varphi(\|\lambda T x_i\|)) \leq [|\lambda|] \Phi(\varphi(\|T x_i\| \leq [|\lambda|] \pi_{\phi,\varphi}(T),$$
$$[|\lambda|] = \inf \{ n \in N \mid \lambda| \leq n \}.$$

Hence

$$\pi_{\phi,\varphi}(\lambda T) \leq [|\lambda|] \pi_{\phi,\varphi}(T).$$

PROPOSITION 1.2.—If

$$T_1 \in L(E, F), \quad T_2 \in \pi_{\phi,\varphi}(F, G).$$

then

$$T_2 T_1 \in \pi_{\phi,\varphi}(E, G)$$

and

$$\pi_{\phi,\varphi}(T_2 T_1) \leq [\|T_1\|]\, \pi_{\phi,\varphi}(T_2).$$

If

$$T_1 \in \pi_{\Phi,\varphi}(E, F), \quad T_2 \in L(F, G),$$

then

$$T_2 T_1 \in \pi_{\phi,\varphi}(E, G)$$

and

$$\pi_{\phi,\varphi}(T_2 T_1) \leq [\|T_2\|] \cdot \pi_{\phi,\varphi}(T_1).$$

Proof.—If

$$T_1 \in L(E, F), \quad T_2 \in \pi_{\phi,\varphi}(F, G)$$

results

$$\pi_{\phi,\varphi}(T_2 T_1) \leq \pi_{\phi,\varphi}[T_2] \cdot \sup_{\|a\| \leq 1} \Phi(\varphi(|\langle \chi_i, T_1^* a \rangle|) \leq$$

$$\leq \pi_{\phi,\varphi}(T_2)[\|T_1\|] \sup_{\|a\| \leq 1} \Phi\left(\varphi\left(\left|\left\langle \chi_i, \frac{T_1^* a}{\|T_1\|}\right\rangle\right|\right)\right).$$

b) $\Phi(\varphi(\|T_2 T_1 \chi_i\|)) \leq [\|T_2\|]\, \Phi(\varphi(\|T_1 \chi_i\|)) \leq$

$$\leq [\|T_2\|]\, \pi_{\phi,\varphi}(T_1) \sup_{\|a\|} \Phi(\varphi(|\langle \chi_i, a \rangle|)).$$

In the similar way that in the papers [2], [3] results

Proposition 1.3.—If F is Banach space

$$\pi_{\phi,\varphi}(E, F)$$

is complete.

Proposition 1.4.—Let

$$\Phi, \Psi \neq \Phi_1, \Phi_\infty$$

be norm functions. Then if

$$T \in \pi_{\Phi, \varphi}(E, F)$$

results

$$T \in \pi_{\Psi^*_{\Phi}, \varphi}(E, P).$$

PROOF. —
For all sequence

$$\} a_i x_i \{ \in E, \quad a_i \in N$$

results

$$\Phi(\varphi(\| T a_i x_i \|)) \leq \pi_{\Phi, \varphi}(T) \sup_{\|a\| \leq 1} \Phi \varphi(| \langle a_i x_i, a \rangle |) \leq$$

$$\leq \pi_{\Phi, \varphi}(T) \sup_{\|a\| \leq 1} \Phi(a_i \cdot \varphi(| \langle x_i, a \rangle |))$$

Or

$$\Phi(a_i \cdot \varphi(\| T x_i \|)) \leq \pi_{\Phi, \varphi}(T) \sup_{\|a\| \leq 1} \Psi(a_i) \cdot \Psi^*_{\Phi}(\varphi(| \langle x_i, a \rangle)),$$

$$\frac{\Phi(a_i \cdot \varphi(\| T x_i \|)}{\Psi(a_i)} \leq \pi_{\Phi, \varphi}(T) \sup_{\|a\| \leq 1} \Psi^*_{\Phi}(\varphi(| \langle x_i) a \rangle |)$$

Hence

$$\Psi^*_{\Phi}(\| T x_i \|) \leq \pi_{\Phi, \varphi}(T) \sup_{\|a\|} \Psi^*_{\Phi}(\varphi(| \langle x_i, a \rangle |) \quad (T \in \pi_{\Psi^*_{\Phi}, \varphi})$$

2.—In this part is generalized the class of (p, r, s) – absolutely summing operators [2] in the following way

DEFINITION 2.1.—An operator $T \in L(E, F)$ is called $(\Phi, \Psi, \chi, \varphi)$- absolutely summing

$$(T \in \pi_{\Phi, \Psi, \chi, \varphi}(E, F))$$

if for all sequences $\{x_i\} \in E$ and $\{b_i\} \in F'$ exists the constant $c \geq 0$ such that

$$\Phi(\varphi(| \langle T x_i, b_i \rangle |)) \leq c \sup_{\|a\| \leq 1} \Psi(\varphi(| \langle x_i, a \rangle |)) \sup_{\|y\| \leq 1} \chi(\varphi(| \langle y, b_i \rangle |))$$

where Φ, Ψ, χ are norm functions such that

$$\chi(x) \geq \Psi^*_\Phi(x), \quad \forall x \in \hat{c}.$$

Remark.—For Φ_p, Ψ_r, χ_s and $\varphi(t) = t$ results the class of $(p, r\ s)$ — absolutely summing operators

$$\left(\frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}\right).$$

The properties of this class are similar to the properties of the class $\pi_{\Phi,\varphi}$. Here we insist to the inclusion relations

Proposition 2.1.—If

$$T \in \pi_{\Phi,\Psi,\chi,\varphi}$$

then

$$T \in \pi_{\bar{\chi}^*_{\bar{\Psi}^*_\Phi},\ \bar{\Psi}^*_\Psi,\ \bar{\chi}^*_\chi,\ \varphi}$$

where $\bar{\Psi}$, $\bar{\chi}$ are new norm functions

$$(\bar{\Psi},\ \bar{\chi} \neq \Phi_1,\ \Phi_\infty).$$

Proof.—

$$\Phi(\varphi(|\langle T x_i, b_i\rangle|)) \leq c \sup_{\|a\|\leq 1} \Psi(\varphi(|\langle x_i, a\rangle|)) \sup_{\|y\|\leq 1} \chi(\varphi((|y, b_i\rangle|))$$

Let be $\{\alpha_i \chi_i\} \in E$ and $\{\beta_i \bar{b}_i\} \in F'$, where $\alpha_i, \beta_i \in N$.
Then

$$\Phi(\varphi(|\langle T \alpha_i \bar{x}_i, \beta_i \bar{b}_i\rangle|)) \leq$$

$$\leq c \sup_{\|a\|\leq 1} \Psi(|\langle \alpha_i \bar{\chi}_i, a\rangle|)) \sup_{\|y\|\leq 1} \chi(\varphi(|\langle y, \beta_i \bar{b}_i\rangle|)) \leq$$

$$\leq c \sup_{\|a\|\leq 1} \bar{\Psi}(\alpha_i) \cdot \bar{\Psi}^*_\Psi(\varphi(|\langle \bar{x}_i, a\rangle|)) \cdot \sup_{\|y\|\leq 1} \bar{\chi}(\beta_i) \cdot \bar{\chi}^*_\chi(\varphi(|\langle y, \bar{b}_i\rangle|))$$

Hence

$$\frac{\Phi(\alpha_i \cdot \beta_i)\,\varphi(|\langle T\bar{x}_i, \bar{b}_i\rangle|)}{\bar{\Psi}(\alpha_i)\cdot\bar{\chi}(\beta_i)} \leq$$

$$\leq c \sup_{\|a\|\leq 1} \bar{\Psi}^*_{\bar{\Psi}}(\varphi(|\langle\bar{x}_i, a\rangle|)) \sup_{\|y\|\leq 1} \bar{\chi}^*_{\chi}(\varphi(|\langle y, \bar{b}_i\rangle|))$$

Or

$$\frac{\bar{\Psi}^*_{\Phi}(\beta_i \cdot \varphi(|\langle T\bar{x}_i, \bar{b}_i\rangle|))}{\bar{\chi}(\beta_i)} \leq$$

$$\leq c \sup_{\|a\|\leq 1} \bar{\Psi}^*_{\bar{\Psi}}(\varphi(|\langle\bar{x}_i, a\rangle|)) \sup_{\|y\|\leq 1} \bar{\chi}^*_{\chi}(\varphi(|\langle y, \bar{b}_i\rangle|))$$

Hence

$$\bar{\chi}^*_{\bar{\Psi}^*_{\Phi}}(\varphi(|\langle T\bar{x}_i, \bar{b}_i\rangle|)) \leq$$

$$\leq c \sup_{\|a\|\leq 1} \bar{\Psi}_{\Psi}(\varphi(|\langle x_i, a\rangle|)) \sup_{\|y\|\leq 1} \bar{\chi}^*_{\chi}(\varphi(\langle y, \bar{b}_i\rangle|)$$

Thus if

$$T \in \pi_{\Phi, \Psi, \chi, \varphi}(E, F)$$

results

$$T \in \pi_{\bar{\chi}^*_{\bar{\Psi}^*_{\Phi}}, \bar{\Psi}^*_{\bar{\Psi}}, \bar{\chi}^*_{\chi}, \varphi}(E, F).$$

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