Available on line at www.rac.es/racsam
Geometry and Topology

REVISTA DE LA REAL ACADEMIA DE CIENCIAS
EXACTAS, FISICAS Y NATURALES.
SERIE A: MATEMATICAS
Madrid (España / Spain)

RACSAM **104** (1), 2010, 87–96. DOI:10.5052/RACSAM.2010.09

On gonality automorphisms of p-hyperelliptic Riemann surfaces

Ewa Tyszkowska

Abstract A compact Riemann surface X of genus $g > 1$ is said to be a p-hyperelliptic if X admits a conformal involution ρ for which X/ρ has genus p. This notion is the particular case of so called cyclic (q, n)-gonal surface which is defined as the one admitting a conformal automorphism δ of order n such that X/δ has genus q. It is known that for $g > 4p + 1$, ρ is unique and so central in the automorphism group of X. We give necessary and sufficient conditions on p and g for the existence of a Riemann surface of genus g admitting commuting p-hyperelliptic involution ρ and (q, n)-gonal automorphism δ for some prime n and we study its group of automorphisms and the number of fixed points of δ. Furthermore, we deal with automorphism groups of Riemann surfaces admitting central automorphism with at most 8 fixed points. The condition on the small number of fixed points of such an automorphism is justified by the study of p-hyperelliptic surfaces.

Sobre automorfismos de gonalidad de superficies de Riemann p-hiperelípticas

Resumen. Una superficie de Riemann compacta X de género $g > 1$ se dice p-hiperelíptica si X admite una involución conforme ρ, tal que X/ρ tiene género p. Las superficies p-hiperelípticas son un caso particular de las superficies (q, n)-gonales cíclicas que se definen como aquellas superficies que admiten un automorfismo conforme δ de orden q y de modo que X/δ tiene género q. En este trabajo nos restringiremos al caso en que q es un número primo mayor que 2. Es un hecho conocido que si $g > 4p+1$, la involución ρ es única y central en el grupo de automorfismos de X. Obtenemos condiciones necesarias y suficientes sobre p y g para la existencia de superficies de Riemann de género g que admiten una involución p-hiperelíptica y un automorfismo (q, n)-gonal que conmutan. Se determina la presentación de un cociente de los grupos de automorfismos de las superficies de Riemann que admiten un automorfismo (q, n)-gonal que sea central y con 8 puntos fijos como máximo. Esta restricción sobre el número de puntos fijos se justifica por el estudio anterior de las superfices que son a la vez p-hiperelípticas y (q, n)-gonales cíclicas.

Submitted by José María Montesinos Amilibia
Received: May 18, 2009. *Accepted*: November 11, 2009
Keywords: Automorphism groups of Riemann surfaces, hyperelliptic Riemann surfaces, p-hyperelliptic Riemann surfaces, p-gonal Riemann surfaces, Fuchsian groups
Mathematics Subject Classifications: 14H, 30F

1 Introduction

A compact Riemann surface X of genus $g \geq 2$ is said to be *p-hyperelliptic* if X admits a conformal involution ρ, called a p-hyperelliptic involution, such that X/ρ is an orbifold of genus p. This notion has been introduced by H. Farkas and I. Kra in [16] where they also proved that for $g > 4p+1$, p-hyperelliptic involution is unique and so central in the group of all automorphisms of X. In [23] it has been proved that every two p-hyperelliptic involutions commute for $3p+2 \leq g \leq 4p+1$ and X admits at most two such involutions if $g > 3p+1$.

In the particular cases $p = 0, 1$, X are called *hyperelliptic* and *elliptic-hyperelliptic* Riemann surfaces respectively. Hyperelliptic Riemann surfaces and their automorphisms have received a good deal of attention in the literature. In [1] and [10] the authors determined the full groups of conformal automorphisms of such surfaces which made possible to classify symmetry types of such actions in [3]. The p-hyperelliptic ($p \geq 1$) surfaces at large have been studied in [4–9, 13–15] and [24], where the most attention has been paid to a study of groups of automorphisms of such surfaces and their symmetries.

In [25], [21] and [22] the classification of conformal actions on p-hyperelliptic Riemann surfaces has been given, up to topological conjugacy, for $p = 0$, 1 and 2, respectively.

A closed Riemann surface X which can be realized as a n-sheeted covering of the Riemann sphere is called n-gonal. Castelnuevo-Severi proved in [11] that if the genus g of X satisfies the inequality $g > (n-1)^2$ then a n-gonality automorphisms group is unique. In [19], Gromadzki justified that for $g \leq (n-1)^2$, X has one conjugacy class of n-gonality automorphism groups in the group $\mathrm{Aut}(X)$ of automorphisms of X. This result has been proved using different techniques by González-Díez in [17]. The authors of [12] found the species of symmetries of real cyclic p-gonal Riemann surfaces while in [2], groups of automorphisms of cyclic trigonal Riemann surfaces have been determined.

A compact Riemann surface X is called (q,n)-gonal if there exists a cyclic group of automorphism C of X, called a (q,n)-gonal group of prime order n such that X/C has genus q. In [18], the conjugacy of (q,n)-gonal groups has been studied. Let us notice that the notion of $(q,2)$-gonality coincides with q-hyperellipticity, whilst $(0,n)$-gonality coincides with n-gonality.

In this paper we study p-hyperelliptic Riemann surface X which admits a conformal automorphism δ, called *(q,n)-gonal automorphism*, of prime order $n > 2$ such that X/δ has genus q [18]. If the genus of X is greater than $4p+1$ then δ and ρ commute. We give necessary and sufficient conditions on p and g for the existence of such a Riemann surface. We show that δ admits 3 or 4 fixed points if $q = 0$; 2–6 if $q = 1$ and at most 8 if $p < q$. We prove that if an automorphism group G of a Riemann surface X has a nontrivial centralizer then there exists a cyclic normal subgroup $H \subseteq G$ and we determine the presentation of a factor group G/H in the case when a central automorphism of X has at most 8 fixed points.

2 Preliminaries

A *Fuchsian group* Λ is a discrete subgroup of the group of linear fractional transformations

$$\mathrm{LF}(2,\mathbb{R}) = \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\},$$

of the complex upper half-plane $\mathcal{H}$ onto itself with compact orbit space. This orbit space can be given an analytic structure such that the projection $\pi_\Lambda \colon \mathcal{H} \to \mathcal{H}/\Lambda$ is holomorphic. The algebraic structure of Λ is determined by the signature $\sigma(\Lambda) = (g; m_1, \ldots, m_r)$, where g, m_i are integers verifying $g \geq 0$, $m_i \geq 2$. The signature determines the presentation of Λ:

generators: $x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g$,
relations : $x_1^{m_1} = \cdots = x_r^{m_r} = x_1 \ldots x_r [a_1, b_1] \ldots [a_g, b_g] = 1$.

Such set of generators is called a *canonical set of generators* and often, by abuse of language, its elements, *canonical generators*. Geometrically x_i are elliptic elements which correspond to hyperbolic rotations and the remaining generators are hyperbolic translations. The integers m_1, m_2,..., m_r are called the *periods* of Λ and g is the genus of the orbit space $\mathcal{H}/\Lambda$. Fuchsian groups with signatures $(g;-)$ are called *surface groups* and they are characterized among Fuchsian groups as these ones which are torsion free.

The group Λ has associated to it a fundamental region F_Λ whose area $\mu(F_\Lambda) = \mu(\Lambda)$, called the *area of the group*, is:

$$\mu(\Lambda) = 2\pi \left(2g - 2 + \sum_{i=1}^{r} (1 - 1/m_i) \right).$$

If Γ is a subgroup of finite index in Λ, then we have the *Riemann-Hurwitz formula* which says that

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}.$$

By Riemann uniformization theorem, each compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group Γ. Furthermore, a group G of automorphisms of a surface $X = \mathcal{H}/\Gamma$ can be represented as $G = \Lambda/\Gamma$ for another Fuchsian group Λ. The number of fixed points of an automorphism of X can be calculated by the following theorem of Macbeath [20].

Theorem 1 *Let $X = H/\Gamma$ be a Riemann surface with the automorphism group $G = \Lambda/\Gamma$ and let $x_1, \ldots, x_r$ be elliptic canonical generators of Λ with periods $m_1, \ldots, m_r$ respectively. Let $\theta\colon \Lambda \to G$ be the canonical epimorphism and for $1 \neq g \in G$ let $\varepsilon_i(g)$ be 1 or 0 according as g is or is not conjugate to a power of $\theta(x_i)$. Then the number $\mathrm{F}(g)$ of points of X fixed by g is given by the formula*

$$\mathrm{F}(g) = |\mathrm{N}_G(\langle g \rangle)| \sum_{i=1}^{r} \varepsilon_i(g)/m_i,$$

where N is a normalizer.

3 p-hyperelliptic Riemann surface with (q, n)-gonal automorphism

In this section we study Riemann surfaces of genera $g > 1$ which are p-hyperelliptic and cyclic (q, n)-gonal simultaneously for a prime $n > 2$ and a natural q. If $g > 4p + 1$, then its (q, n)-gonal automorphism and p-hyperelliptic involution commute. The first theorem gives necessary and sufficient conditions on p and g for the existence of such a surface.

Theorem 2 *There exists a p-hyperelliptic Riemann surface of genus $g \geq 2$ admitting (q, n)-gonal automorphism commuting with a p-hyperelliptic involution if and only if $p = n\gamma + b(n-1)/2$ and $g = nq + a(n-1)/2$ for some integers γ, b, a such that*

$$b = -2 \text{ or } b \geq 0, \qquad b \leq a \leq 2(b+1), \qquad 0 \leq \gamma \leq (q+1)/2. \tag{1}$$

Furthermore, the (q, n)-gonal automorphism admits $a + 2$ fixed points.

PROOF. Assume that a Riemann surface $X = \mathcal{H}/\Gamma$ admits p-hyperelliptic involution ρ and (q, n)-gonal automorphism δ. The groups $\langle \delta \rangle$ and $\langle \rho \rangle$ can be identified with Γ_δ/Γ and Γ_ρ/Γ, where Γ_δ and Γ_ρ are Fuchsian groups containing Γ as a normal subgroup of index n and 2, respectively. By the Riemann-Hurwitz formula they have signatures

$$\sigma(\Gamma_\delta) = (q; n \overset{r}{\ldots}, n) \quad \text{and} \quad \sigma(\Gamma_\rho) = (p; 2, \overset{s}{\ldots}, 2), \tag{2}$$

where $s = 2g + 2 - 4p$ and $r = 2 + (2g - 2nq)/(n-1)$. Thus $g = nq + a(n-1)/2$ for $a = r - 2$. If ρ and δ commute then they generate the group $\mathbb{Z}_{2n}$ which can be represented by Λ/Γ for a Fuchsian group Λ with the signature

$$(\gamma; 2, \overset{k_1}{\ldots}, 2, n, \overset{k_2}{\ldots}, n, 2n, \overset{k_3}{\ldots}, 2n). \tag{3}$$

By the Riemann-Hurwith formula

$$2g - 2 = 4n\gamma - 4n + nk_1 + 2k_2(n-1) + k_3(2n-1) \tag{4}$$

and according to Theorem 1

$$nk_1 = s - k_3, \qquad 2k_2 = r - k_3.$$

By substituting the last equalities to (4), we obtain $p = n\gamma + b(n-1)/2$, for an integer b such that $a = 2b + 2 - k_3$. Thus

$$k_1 = 2q + a - 4\gamma - 2b, \qquad k_2 = a - b, \qquad k_3 = 2 + 2b - a$$

are nonnegative integers if and only if the inequalities (1) are satisfied.

Conversely, assume that $g = nq + a(n-1)/2$ and $p = n\gamma + b(n-1)/2$ for some integers a, b and γ satisfying the inequalities (1). Then there exists a Fuchsian group Λ with the signature (3). Let $\theta\colon \Lambda \to \langle\rho\rangle \oplus \langle\delta\rangle$ be an epimorphism which maps all hyperbolic generators of Λ onto $\rho\delta$, the first k_1 of elliptic generators onto ρ and the remaining in the following way :

$$\underbrace{\delta\ldots\delta}_{(k_2+1)/2}\ \underbrace{\delta^{-1}\ldots\delta^{-1}}_{(k_2-3)/2}\ \delta^{-2}\ \underbrace{\rho\delta\ldots\rho\delta}_{(k_3+1)/2}\ \underbrace{\rho\delta^{-1}\ldots\rho\delta^{-1}}_{(k_3-3)/2}\ \rho\delta^{-2} \quad \text{if } k_2 \equiv 1\ (2) \text{ and } k_3 \equiv 1\ (2),$$

$$\underbrace{\delta\ldots\delta}_{(k_2+1)/2}\ \underbrace{\delta^{-1}\ldots\delta^{-1}}_{(k_2-3)/2}\ \delta^{-2}\ \underbrace{\rho\delta\ldots\rho\delta}_{k_3/2}\ \underbrace{\rho\delta^{-1}\ldots\rho\delta^{-1}}_{k_3/2} \quad \text{if } k_2 \equiv 1\ (2) \text{ and } k_3 \equiv 0\ (2),$$

$$\underbrace{\delta\ldots\delta}_{k_2/2}\ \underbrace{\delta^{-1}\ldots\delta^{-1}}_{k_2/2}\ \underbrace{\rho\delta\ldots\rho\delta}_{(k_3+1)/2}\ \underbrace{\rho\delta^{-1}\ldots\rho\delta^{-1}}_{(k_3-3)/2}\ \rho\delta^{-2} \quad \text{if } k_2 \equiv 0\ (2) \text{ and } k_3 \equiv 1\ (2),$$

$$\underbrace{\delta\ldots\delta}_{k_2/2}\ \underbrace{\delta^{-1}\ldots\delta^{-1}}_{k_2/2}\ \underbrace{\rho\delta\ldots\rho\delta}_{k_3/2}\ \underbrace{\rho\delta^{-1}\ldots\rho\delta^{-1}}_{k_3/2} \quad \text{if } k_2 \equiv 0\ (2) \text{ and } k_3 \equiv 0\ (2).$$

Then the kernel of θ is a surface Fuchsian group Γ of genus g while $\theta^{-1}(\rho)$ and $\theta^{-1}(\delta)$ are Fuchsian groups with the signatures (2). Thus $\mathcal{H}/\Gamma$ is a p-hyperelliptic Riemann surface admitting (q,n)-gonal automorphism. It is easy to notice that for $k_2 < 3$ or $k_3 < 3$, such an epimorphism does not exist if and only if $k_2 + k_3 + \gamma = 0$ or $k_2 + k_3 = 1$. The first equality is never satisfied since if $k_2 + k_3 = 0$ then $b = -2$ and $p = n(\gamma - 1) + 1$ what requires $\gamma \geq 1$. The second one occurs for $b = -1$ and therefore this value of b is rejected. ■

Corollary 1 *Let X be a p-hyperelliptic Riemann surface of genus $g > 4p+1$. Then for any prime $n \geq 3$,*

(i) *X can be realized as cyclic n-sheeted covering of the Riemann sphere if and only if $p = 0$ and $g = n - 1$ or $g = (n-1)/2$ and its cyclic n-gonal automorphism admits 4 or 3 fixed points, respectively.*

(ii) *X can be realized as cyclic n-sheeted covering of an elliptic curve if and only if $p = 0$ and $g \in \{2n-1, (3n-1)/2, n\}$ or $p = (n-1)/2$ and $g \in \{3n-2, (5n-3)/2\}$ and its $(1,n)$-gonal automorphism admits 4, 3, 2 or 6, 5 fixed points, respectively.*

Corollary 2 *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface of genus $g \geq 2$ which admits p-hyperelliptic involution ρ and (q,n)-gonal automorphism δ for $p < n$. If δ and ρ commute then $p = b(n-1)/2$, $g = nq + a(n-1)/2$ for integers a, b in range $0 \leq b \leq 2$ and $b \leq a \leq 2b+2$ and a Fuchsian group Λ such that $\langle\delta,\rho\rangle = \Lambda/\Gamma$ has a signature $(0; 2, \overset{2q+a-2b}{\ldots}, 2, n, \overset{a-b}{\ldots}, n, 2n, \overset{2b+2-a}{\ldots}, 2n)$. Furthermore, δ admits $a + 2 \leq 8$ fixed points.*

The last corollary is the inspiration for the next section in which we study the groups of automorphisms of a Riemann surface admitting a central automorphism with at most 8 fixed points.

4 Automorphism groups of a Riemann surface with nontrivial centralizer

Let G be an automorphism group of a Riemann surface X of genus $g \geq 2$ admitting a central element δ of order n. If $z \in X$ is a fixed point of δ, then δ preserves all points in the orbit Gz. Assume that the stabilizer $\mathrm{Stab}(z)$ of z is a cyclic group of order m generated by $x \in G$. Then n divides m and $\langle \delta \rangle = \langle x^{m/n} \rangle$. Any element $g \in G$ permutes points of Gz and we shall assign a permutation $\sigma_g \in S_k$ to g, where $k = |Gz| = |G|/m$. The permutation σ_x splits into product of cycles of lengths $t_1, \ldots, t_\beta$, respectively, where t_j divide m. Let $g_1, \ldots, g_\beta$ be different elements of G for which t_j are the smallest positive integers such that $x^{t_j} \in g_j \langle x \rangle g_j^{-1}$. Then

$$Gz = \{h_1 z, \ldots, h_\alpha z, g_1 z, x g_1 z, \ldots, x^{t_1 - 1} g_1 z, \ldots, g_\beta z, \ldots, x^{t_\beta - 1} g_\beta z\},$$

where $\alpha = k - (t_1 + \cdots + t_\beta)$ and $h_i \in G$ normalize $\langle x \rangle$. We shall denote points $h_i z$ by z_i, in particular z by z_1, and points $x^l g_j z$ by $z_{j,l}$. In order to determine the presentation of G we shall need the following lemmata.

Lemma 1 *Let r_i be the smallest positive integer such that $g_i^{r_i} \in \langle x \rangle$ for $i = 1, \ldots, \beta$. Then there exists an integer b_i such that $b_i \equiv 1\ (n)$, $(m/t_i, b_i) = 1$, $b_i^{r_i} \equiv 1\ (m/t_i)$ and*

$$g_i x^{t_i} g_i^{-1} = x^{b_i t_i}.$$

Moreover, $g_i^{r_i} = x^{p_i}$ for some p_i such that $p_i \equiv 0\ (t_i)$ and $b_i \equiv 1\ (m/\gcd(m, p_i))$.

PROOF. Assume that $x^{t_i} = g_i x^{t_i l_i} g_i^{-1}$ for an integer l_i co-prime with m/t_i. Then there exist a_i and b_i such that $a_i m/t_i + b_i l_i = 1$ and so $g_i x^{t_i} g_i^{-1} = x^{b_i t_i}$.

If c is an integer such that x^c and g_i commute then $c \equiv 0\ (t_i)$ what implies $b_i \equiv 1\ (m/\gcd(m, c))$. Otherwise, a smaller power than x^{t_i} would belong to $g_i \langle x \rangle g_i^{-1}$. In particular, $p_i \equiv 0\ (t_i)$, $b_i \equiv 1\ (m/\gcd(m, p_i))$ and $b_i \equiv 1\ (n)$. Finally, since $g_i^{r_i}$ and x commute, it follows that $b_i^{r_i} \equiv 1\ (m/t_i)$. ■

Lemma 2 *For any i in range $1 \leq i \leq \beta$, g_i maps the set $F = \{z_1, \ldots, z_\alpha\}$ into $Gz \setminus F$. Furthermore, if g_i maps a point of F into $z_{i',l}$ for some $1 \leq i' \leq \beta$ and $1 \leq l \leq t_{i'}$ then $t_i = t_{i'}$ and $g_{i'}$ maps a point of F into $z_{i,-l}$.*

PROOF. On a contrary, suppose that $g_i(z_j) = z_{j'}$ for some $z_j, z_{j'} \in F$. Then z_j is a fixed point of $g_i^{-1} x g_i$. Thus $g_i^{-1} x g_i \in h_j \langle x \rangle h_j^{-1} = \langle x \rangle$ what implies $z_{i,0} = z_{i,1}$, a contradiction. So g_i maps every $z_j \in F$ into some point $z_{i'l} \in Gz \setminus F$. Thus $x^l g_{i'} x g_{i'}^{-1} x^{-l} = g_i h_j x h_j^{-1} g_i^{-1} \in g_i \langle x \rangle g_i^{-1}$ what implies $t_i = t_{i'}$.

Now let $g \in G$ be such an element that $g_{i'}(gz) = z_{i,-l}$ Then $z_{i',0} = g_{i'} g(g^{-1} z) = x^{-l} g_i x^s (g^{-1} z)$ for some integer s and so $z_{i',l} = g_i(x^s g^{-1} z)$ what implies $g^{-1} z = z_j$. Thus $g x g^{-1} \in \langle x \rangle$ what means that $gz \in F$. ■

By the proof of Lemma 2, we obtain the following

Corollary 3 *If $\beta \neq 0$ then $\alpha \leq t_1 + \cdots + t_\beta$ and G is generated by x and $g_1, \ldots, g_\beta$.*

Lemma 3 *If $g_s(z_{i_0, l_0}) = z$ for some s, i_0 and l_0 in range $1 \leq s, i_0 \leq \beta$ and $1 \leq l_0 \leq t_{i_0}$, respectively, then $t_s = t_{i_0}$. In particular, for $s = i_0$, the element $g = g_{i_0} x^{l_0 - 1}$ satisfies the relation $(gx)^2 = 1$ modulo $x^{t_{i_0}}$ and*

$$g(z_{i,l}) = z_{i',l'} \quad \text{if and only if} \quad g(z_{i',l'+1}) = z_{i,l-1}, \tag{5}$$

$$g(z_j) = z_{i,l} \quad \text{if and only if} \quad g(z_{i,l+1}) = z_j, \tag{6}$$

$$\text{if } g(z_{i,l+1}) = z_{i,l} \quad \text{then } t_i \text{ is even and for } i = i_0, \quad x^{t_i/2} g = g x^{1-l} g x^l. \tag{7}$$

PROOF. Since $g_s x^{l_0} g_{i_0} \in \langle x \rangle$, it follows that $g_s^{-1} x g_s = x^{l_0} g_{i_0} x g_{i_0}^{-1} x^{-l_0}$ what implies $t_{i_0} = t_s$. If $s = i_0$ and $g = g_{i_0} x^{l_0 - 1}$ then $(gx)^2 z = z$ and so $(gx)^2 = x^q$ for some integer q. Thus $g^2 x = g x^{q-1} g^{-1}$. On the other hand $x^q = g x g^{-1} g^2 x$ implies that $g^2 x = g x^{-1} g^{-1} x^q$. Consequently, $g x^q g^{-1} = x^q$ and so $q \equiv 0 \ (t_{i_0})$.

The statements (5) and (6) follow from the relation $(gx)^2 = x^q$.

If $g(z_{i,l+1}) = z_{i,l}$ then gx preserves point $z_{i,l}$ and so $gx = x^l g_i x^r g_i^{-1} x^{-l}$ for some r not being a multiple of t_i. If t_i is odd then rising the last equation to second power we obtain that $g x^{r'} g^{-1} \in \langle x \rangle$ for some integer $r' < t_i$ against our choice of t_i. For even t_i, $r = t_i/2$ and additionally if $i = i_0$ then using the relation $(gx)^2 = x^q$ we obtain $g x^{1-l} g x^l = x^{t/2} g$. ■

Lemma 4 *Let $i, j \in \{1, \ldots \beta\}$, $l \in \{1, \ldots, t_j\}$ and $l' \in \{1, \ldots, t_i\}$.*

(i) *If $g \in G$ preserves point $z_{j,l}$ then $g^{t_j} \in \langle x \rangle$.*

(ii) *If $g_i(z_{j,l}) = z_{i,l'}$, then t_j divides t_i.*

PROOF. (i) By the assumption, $g \in x^l g_j \langle x \rangle g_j^{-1} x^{-l}$ and so $g^{t_j} \in \langle x \rangle$.

(ii) Here $g_j = x^{-l} g_i^{-1} x^{l'} g_i x^s$ for some integer s. Thus $g_j x^{t_i} g_j^{-1} \in \langle x \rangle$ and so t_j divides t_i. ■

Theorem 3 *Let G be a group of automorphisms of a Riemann surface X admitting a central automorphism δ of order n and suppose that δ admits $k \leq 8$ fixed points in the same orbit. Then for $k > 1$, there exists an element $x \in G$ of order $m = |G|/k$ and an integer t dividing m such that $H = \langle x^t \rangle$ is a normal subgroup of G, $\delta \in H$ and G/H has one of presentations listed in Table 1. For $k = 1$, G is a cyclic group.*

PROOF. Since $k < 9$ then the sequence of parameters for the action of G on such an orbit must be of the form $C_k = (\alpha; t_1, t_2, t_3)$. First we show that some sequences are not possible. For, suppose that $t_1 \neq t_2$ and $t_1 \neq t_3$. Then by Lemma 3, $g_1(z_{1,l_0}) = z$ for some $l_0 = 1, \ldots, t_1$ and we shall use $g = g_1 x^{l_0 - 1}$ instead of g_1. Furthermore, according to Lemma 2, $g(F) \subset \{z_{1,0}, \ldots z_{1,t_1}\}$ and if $z_{1,l}$ is an image of a point from F then $z_{1,-l}$ is also an image of a point from F. In particular, if F contains only two points $z_1 = z$ and z_2 then $x^{-l} g z = g(z_2) = x^l g z$ what requires t even. Thus the sequences $C_5 = (2; 3, 0, 0)$, $C_7 = (2; 3, 2, 0)$ and $C_7 = (2; 5, 0, 0)$ must be rejected. For $C_8 = (3; 3, 2, 0)$, without lost of generality we can assume that $g(z_2) = xgz$ and $g(z_3) = x^2 g z$. Thus by (6), $z_2 = g(x^2 g z)$ and $z_3 = g(gz)$. So it remains that g preserves or exchanges points $z_{2,0}$ and $z_{2,1}$ what by item (i) of Lemma 4 implies that $g^2 \in \langle x \rangle$ or $gx = g_2 x g_2^{-1} = x^{-1} g$, respectively. Thus not all points in Gz are different against the assumption. Similarly for $C_8 = (3; 5, 0, 0)$, we can assume that $g(z_2) = x^2 g z$ and $g(z_3) = x^3 g z$. Thus by (6) and (7), $\sigma_g = (1, 4, 5)(2, 6, 8, 3, 7)$ and so $g^3 \in \langle x \rangle$. However $g^3(z_2) \neq z_2$, a contradiction once again.

If t_i does not divide t_1 for $i = 1$ or 2 then by item (ii) of Lemma 4, $g(z_{i,l}) \notin \{z_{1,1}, \ldots, z_{1,t_1}\}$. Thus for $C_8 = (1; 5, 2, 0)$ and $C_6 = (1; 3, 2, 0)$, g preserves points $z_{2,0}$, $z_{2,1}$ or exchanges them what has been shown is impossible. Using (7) for $C_8 = (1; 3, 2, 2)$, we conclude that σ_g is a product of cycles, one of which is $(1, 2, 3)$, and so $g^3 = x^p$ for some integer p. However since σ_g neither preserves nor exchanges points $z_{i,0}$ and $z_{i,1}$, it follows that $g^3(z_{2,0}) \neq z_{2,s}$ for $s = 0, 1$, a contradiction. The sequence $C_8 = (1; 4, 3, 0)$ is also impossible since there does not exist σ_g for which $g(z_{2,l}) \neq z_{1,l'}$ and $g(z_{2,l+1}) \neq z_{2,l}$ for $l = 0, 1, 2$ and $l' = 1, \ldots, 4$.

Since the case $(1; 3, 2, 2)$ is rejected and $k < 9$, it follows that two parameters t_i in the sequence $C_k = (\alpha; t_1, t_2, t_3)$ can be equal if and only if $t_i \in \{0, 2\}$ or $t_i \in \{0, 3\}$ for $i = 1, 2, 3$. We shall describe only the first possibility since the second one can be solved in the similar way. However in most cases all parameters t_1, t_2 and t_3 are different and first we concentrate on them. So assume that t_1, t_2, t_3 are different integers. Then by Lemma 3, there exist i and l in range $1 \leq i \leq 3$ and $1 \leq l \leq t_i$, respectively such that $g_i(z_{i,l}) = z$ and it is convenient to exchange g_i for $g = g_i x^{l-1}$ which satisfies the relation $(gx)^2 \equiv 1 \ (x^t)$, for $t = t_i$. From now on we will write all relations modulo x^t unless we say differently. Let us notice that $g(x g^s z) = x^{-1} g^{s-1} z$ for $s = 1, \ldots, r$ and so $g(xgz) = z$. We find the permutation σ_g and by

k	Case	Presentation of $\tilde{G}$
$2 \leq k \leq 8$	$k.1$	$\langle g : g^k = 1\rangle$
	$k.2$	$\langle x, g : x^2 = 1, g^k = 1, (gx)^2 = 1\rangle$
4	4.3	$\langle x, g : x^3 = 1, g^3 = 1, (gx)^2 = 1\rangle$
5	5.3	$\langle x, g : g^4 = 1, gxg^{-1} = xgx^{-1}, g^2 = x^2(gx)x^{-2}\rangle$
6	6.3	$\langle x, g : x^4 = 1, g^3 = 1, (gx)^2 = 1\rangle$
	6.4	$\langle x, g : x^3 = 1, g^6 = 1, xg^3x^{-1} = gx\rangle$
	6.5	$\langle x, g : x^2 = 1, g^3 = 1, (gx)^3 = 1\rangle$
7	7.3	$\langle x, g : g^3 = 1, x^3gx^{-3} = gx^2g^{-1}, gx^3g^{-1} = x^2(gx)x^{-2}\rangle$
8	8.3	$\langle x, g : x^3 = 1, g^4 = 1, (gx)^2 = 1\rangle$
	8.4	$\langle x, g : g^6 = 1, gxg^{-1} = xg^2x^{-1}, (gx)^2 = 1\rangle$
	8.5	$\langle x, g : x^4 = 1, g^8 = 1, (gx)^2 = 1, [g^2, x] = 1\rangle$
	8.6	$\langle x, g : g^7 = 1, (gx)^2 = 1, x^2g^{-1}x^{-2} = gxg^{-1}\rangle$
	8.7	$\langle x, g : x^3 = 1, g^4 = 1, (gx)^3 = 1, [g^2, x] = 1\rangle$
	8.8	$\langle x, g : x^3 = 1, g^4 = 1, (gx)^3 = g^2, [g^2, x] = 1\rangle$
	8.9	$\langle x, g : x^3 = 1, g^3 = 1, (gx)^2 = g^{-1}xg\rangle$
	8.10	$\langle x, g : g^3 = 1, (gx)^4 = 1, xgx^{-1} = gx^{-1}g^{-1}\rangle$
	8.11	$\langle x, g : x^3 = 1, g^7 = 1, gx = g^{-1}xg\rangle$
	8.13	$\langle x, g : x^2 = 1, g^4 = 1, (gx)^4 = 1, [g^2, x] = 1\rangle$
	8.14	$\langle x, g : x^2 = 1, g^8 = 1, (gx)^8 = 1, [g^2, x] = 1\rangle$
	8.15	$\langle x, g : x^4 = 1, g^4 = 1, (gx)^2 = 1, [g^2, x^2] = 1\rangle$
	8.16	$\langle x, g_1, g_2 : x^2 = (g_1x)^2 = g_1^4 = (g_2x)^2 = 1, g_1^2 = g_2^2\rangle$

Table 1. The presentation of the group G/H

consideration how it acts on points of Gz we obtain relations which determine the presentation of G. We consider the case with $t_1 = t = 4$ as an example, the remaining cases can be solved in the similar way. First we find the all possible values of g^2z. If $g^2 = xgz$ then $g^3 \in \langle x\rangle$ and by (5), $g(x^2gz) = x^3gz$. Using the relation $(gx)^2 = 1$ and $g^3 = 1$ we calculate that $(gx^3g)x(gx^3g)^{-1} = x^{-1}$ what means that $g(x^3gz)$ is a fixed point of x, say z_2. Thus by (6), $g(z_2) = x^2gz$. It is easy to notice that Gz cannot have any other points but z, z_2, gz, …, x^3gz since otherwise we get a contradiction with lemata. So we get the sequence $C_6 = (2, 4, 0, 0)$ for which $\sigma_g = (1, 3, 4)(2, 5, 6)$. By Lemma 1 and Corollary 3, G is generated by x, g and admits a normal cyclic subgroup $H = \langle x^4\rangle$. By analyzing σ_g we conclude that $\tilde{G} = G/H$ has the presentation 6.3.

Next suppose that $g^2 = x^2gz$. Then by (5), x^3gz is a fixed point of g and so by Lemma 4, $g^4 \in \langle x\rangle$. Thus $g(x^2gz) = g^3z = xgz$. Since Gz cannot have any additional points, it follows that $C_5 = (1; 4, 0, 0)$, $\sigma_g = (1, 2, 4, 3)$ and $\tilde{G}$ has the presentation 5.3.

If $g^2z = x^3gz$ then $gxg^2z = g^2z$. Thus gx preserves point g^2z and so $gx = g^2x^2g^{-2}$. Consequently $z = gxgz = g^2x^2g^{-1}z = g^2x^3gz = g^4z$. So $g^4 \in \langle x\rangle$ and we conclude that for $C_5 = (1; 4, 0, 0)$, $\sigma_g = (1, 2, 5, 3)$ and $\tilde{G}$ has the presentation $\tilde{G} = \langle x, g : x^4 = 1, g^4 = 1, gxg^{-1} = x^2gx^{-2}, gx^2g^{-1} = xg\rangle$ which is isomorphic to 5.3.

If $g^2z = z_2 \in F$ then $g(z_2) = x^3gz$. Thus according to Lemma 2, there exists $z_3 \in F$ such that $xgz = g(z_3)$ and so by (6), $z_3 = g(x^2gz)$. If $g(x^3gz) = x^2gz$ then by (7), $x^2g = gx^3gx^2$. However $x^2g(gz) \neq gx^3gx^2(gz)$ and so there exists one more point $z_4 \in F$ such that $g(x^3gz) = z_4$. Thus $g(z_4) =$

x^2gz and $\sigma_g = (1,5,2,8,4,7,3,6)$. So for $C_8 = (4;4,0,0)$, $\tilde{G}$ has the presentation 8.5.

Finally suppose that $g^2z = z_{2,0}$. Then $g(z_{2,1}) = z_{1,3}$ and so by item (ii) of Lemma 4, $t_2 = 2$. Let us consider all possible values of g^3z. If $g^3z = z_{1,1}$ then $g^4 \in \langle x\rangle$ and $z_{2,1} = g(z_{1,2})$. Furthermore, $g(z_{1,3}) \neq z_{1,2}$ since otherwise by (7), $x^2g = gx^3gx^2$. However by evaluation the last equality in $z_{1,0}$ we obtain different points. Thus there exists $z_2 \in F$ such that $g(z_{1,3}) = z_2$ and consequently $g(z_2) = z_{1,2}$. So for $C_8 = (2;4,2,0)$, $\sigma_g = (1,3,7,4)(2,5,8,6)$ and $\tilde{G}$ has the presentation 8.15.

If $g^3z = z_{1,2}$ then $z_{2,1} = g(z_{1,3})$ and it remains that $g(z_{1,2}) = z_{1,1}$ or $g(z_{1,2}) = z_2$ for some $z_2 \in F$. In the first case by (7), $x^2g = g^2x$ against the assumption that $z_{2,0} = g^2z$. The second one is also impossible since then $g(z_2) = z_{1,1}$. However there does not exist an integer s such that $g^2x^2g(z_2) = xgx^s(z_2)$.

If $g^3 = z_{2,1}$ then $g(z_{1,2}) \neq z_{1,1}$ and g does not preserve $z_{1,2}$. Thus there exists $z_2 \in F$ such that $g(z_{1,2}) = z_2$ what implies $g(z_2) = z_{1,1}$. So it remains that $g(z_{1,3}) = z_{1,2}$. However $x^2g(z_2) \neq gx^3gx^2(z_2)$, a contradiction with (7).

Now we shall consider the sequences $C_k = (\alpha;t_1,t_2,t_3)$, where $t_i \in \{0,2\}$ for $i = 1,\,2,\,3$. First suppose that one of g_i, say g_1, satisfies $(g_ix)^2 \in \langle x\rangle$. Then $xg_1^s = g_1^{-s}x$ for $s = 1,\,\ldots,\,r$, where r is the smallest positive integer such that $g_1^r \in \langle x\rangle$. Thus g_1^sz is a fixed point of x if and only if r is even and $s = r/2$, in this case we shall denote the point $g_1^{r/2}z$ by z_2. In particular, if $r = k$ then $\alpha = 1$ or 2 according to k being odd or even, respectively, $g = g_1$ and x generate G and

$$\tilde{G} = \langle x, g : x^2 = 1, g^k = 1, (gx)^2 = 1\rangle. \tag{8}$$

Since g_1 neither preserves nor exchanges points $z_{j,l}$ and $z_{j,l+1}$ for $j = 1,\,2,\,3$ and $l = 0,\,1$, it follows that we have the following possibilities for $r < k$:

(i) $r = 3$, $C_6 = (2;2,2,0)$, $\sigma_{g_1} = (1,3,4)(2,5,6)$,

(ii) $r = 4$, $C_8 = (2;2,2,2)$, $\sigma_{g_1} = (1,3,2,4)(5,7)(6,8)$ or $(1,3,2,4)(5,6,7,8)$,

(iii) $r = 4$, $C_8 = (4;2,2,0)$, $\sigma_{g_1} = (1,5,2,6)(3,8,4,7)$. By analyzing σ_{g_1} we conclude that G is generated by x and g_2. So we shall find σ_{g_2} in order to determine the presentation of G. If $z = g_2(z_{1,l})$ for some $l \in \{0,1\}$ then not all points in Gz are different. So we can assume that $(g_2z)^2 \in \langle x\rangle$.

(i) Since xg_1 preserves point $z_{2,0}$, it follows that $xg_1 = g_2xg_2^{-1}$. Thus $g_1 = g_2^{-2}$ and so $g_2^6 \in \langle x\rangle$. Consequently $\tilde{G}$ has the presentation (8), where $k = 6$ and $g = g_2$.

(ii) Let us notice that the first permutation leads to a contradiction. Indeed, since g_1^2 preserves $z_{2,0}$, it follows that $g_1^2 = g_2xg_2^{-1}$. Thus if z' is a fixed point of g_1^2 then $g_2^{-1}(z') \in F$. However g_1^2 admits 4 fixed points and therefore not all points in Gz are different. By the second permutation, xg_1 preserves $(z_{2,0})$, what implies $g_1 = xg_2xg_2^{-1} = g_2^{-2}$. Thus $g_2^8 \in \langle x\rangle$ and so $\tilde{G}$ has the presentation (8), where $k = 8$ and $g = g_2$.

(iii) Since xg_1^2 preserves $z_{2,0}$, it follows that $xg_1^2 = g_2xg_2^{-1}$ and so $g_1^2 = g_2^2$. Thus we conclude that $\sigma_{g_2} = (1,7,2,8)(3,5,4,6)$ and $\tilde{G}$ has the presentation 8.16.

Next suppose that $(g_ix)^2 \notin \langle x\rangle$ for $i = 1,\,2,\,3$. Then without lost of generality we can assume that $z_{2,l} = x^lg^{-1}z$ for $l = 0,\,1$ and $g = g_1$. Let us notice that $g(z_{2,1}) \neq z_{1,1}$ since otherwise $gxg^{-1} = xgx^s$ for some integer s and evaluation the last equality in $z_{1,0}$ implies that $g(z_{1,s}) = z_{1,1}$, a contradiction. Since g does not preserve any points $z_{i,l}$ and $g(z_{2,l}) \neq z_{2,l+1}$ for $i = 1,\,2,\,3$, it follows that the sequences $C_5 = (1;2,2,0)$ and $C = (3;2,2,0)$ are impossible. For $C_6 = (2;2,2,0)$, $g(z_{2,1}) = z_2$ and $g(z_2)$ is one of points $z_{1,1}$, $z_{2,0}$, $z_{2,1}$. Using Lemma 2 we check that all possibilities provide a contradiction except the first one. Here $\sigma_g = (1,3,5)(2,4,6)$ and we conclude that $\tilde{G}$ has the presentation 6.5. For $C_8 = (2;2,2,2)$ we obtain the presentation 8.12. Finally for $C_8 = (4;2,2,0)$, since $g(F) = Gz \setminus F$, we can assume that $z_2 = g^2z$ and so $[g^2,x] = 1$. Furthermore, $g^3z \in \{z_{1,1}, z_{2,0}, z_{2,1}\}$. If $g^3z = z_{1,1}$ then $z_{1,0} = xg^3z = g^2xgz$ what implies $z_{1,1} = z_{2,0}$, a contradiction. If $g^3z = z_{2,0}$ then $g^4 \in \langle x\rangle$ and so $g^2(xgz) = xg^3z$ and $g^2(xg^3z) = xgz$. Thus $\sigma_g = (1,5,2,7)(3,8,4,6)$ and $\tilde{G}$ has the presentation 8.13. If $g^3z = z_{2,1}$ then $g^{-1}z = xg^3z = g(gxgz)$. Here $\sigma_g = (1,5,2,8,4,6,3,7)$ and $\tilde{G}$ has the presentation 8.14.

If $\beta = 0$, then G is generated by two elements g and x, $\langle x \rangle$ is a normal subgroup of G and $\tilde{G} = \langle g : g^k = 1 \rangle$. ■

By corollaries 1 and 3 we obtain the following

Corollary 4 *Let X be a p-hyperelliptic Riemann surface with a central (q, n)-gonal automorphism δ. Then for $p < n$ or $q = 0, 1$, δ has at most 8 fixed points and an automorphism group of X is determined by Theorem 3.*

Acknowledgement. The author supported by the Research Grant N. N201 366436 of the Polish Ministry of Sciences and Higher Education.

References

[1] Brandt, R. and Stichtenoth, H., (1986). Die Automorphismengruppen hyperelliptischer Kurven, *Manuscripta Math.*, **55** ,1, 83–92. DOI: 10.1007/BF01168614

[2] Bujalance, E.; Cirre, F. J. and Gromadzki, G., (2009). Groups of automorphisms of cyclic trigonal Riemann surfaces, *J. Algebra*, **322**, 4, 1086–1103. DOI: 10.1016/j.jalgebra.2009.05.017

[3] Bujalance, E.; Cirre, F. J.; Gamboa, J. M. and Gromadzki, G., (2001). Symmetry types of hyperelliptic Riemann surfaces, *Mém. Soc. Math. Fr. (N.S.)*, **86**.

[4] Bujalance, E. and Costa, A. F., (1997). On symmetries of p-hyperelliptic Riemann surfaces, *Math. Ann.*, **308**, 1, 31–45. DOI: 10.1007/s002080050062

[5] Bujalance, E. and Etayo, J. J., (1988). Large automorphism groups of hyperelliptic Klein surfaces, *Proc. Amer. Math. Soc.*, **103**, 3, 679–686. DOI: 10.2307/2046834

[6] Bujalance, E. and Etayo, J. J., (1988). A characterization of q-hyperelliptic compact planar Klein surfaces, *Abh. Math. Sem. Univ. Hamburg*, **58**, 1, 95–102. DOI: 10.1007/BF02941371

[7] Bujalance, E.; Etayo, J. J. and Gamboa, J. M., (1986). Surfaces elliptiques-hyperelliptiques avec beaucoup d'automorphismes, *C. R. Acad. Sci. Paris Sér. I Math.*, **302**, 10, 391–394

[8] Bujalance, E.; Etayo, J. J. and Gamboa, J. M., (1987). Topological types of p-hyperelliptic real algebraic curves. *Math. Z.*, **194**, 2, 275–283.

[9] Bujalance, E., Etayo, J. J., Gamboa, J. M. and Gromadzki, G., (1990). *Automorphisms Groups of Compact Bordered Klein Surfaces. A Combinatorial Approach*, Lecture Notes in Math., **1439**, Springer Verlag.

[10] Bujalance, E.; Gamboa, J. M. and Gromadzki, G., (1993). The full automorphisms group of hyperelliptic Riemann surfaces, *Manuscripta Math*, **79**, 1, 267–282. DOI: 10.1007/BF02568345

[11] Castelnuevo, G., (1906). Sulle serie algebriche di gruppi di punti apparteneti ad una curve algebraica, *Rendiconti della R. Academia dei Lincei*, Series **5**, XV, (memorie scelte p. 509).

[12] Costa, Antonio F. and Izquierdo, Milagros, (2004). *Symmetries of real cyclic p-gonal Riemann surfaces, Pacific J. Math.*, **213**, 2, 231–243. DOI: 10.2140/pjm.2004.213.231

[13] Estrada, B., (2000). Automorphism groups of orientable elliptic-hyperelliptic Klein surfaces, *Ann. Acad. Sci. Fenn. Math.*, **25** , 439–456.

[14] Estrada, B., (2002). Geometrical characterization of p-hyperelliptic planar Klein surfaces, *Comput. Methods Funct. Theory*, **2**, 1, 267–279.

[15] Estrada, B. and Martínez, E., (2001). On q-hyperelliptic k-bordered tori, *Glasg. Math. J.*, **43**, 3, 343–357. DOI: 10.1017/S0017089501030142

[16] FARKAS, H. M. AND KRA, I., (1980). *Riemann Surfaces*, Graduate Text in Mathematics, Springer-Verlag.

[17] GONZÁLEZ-DÍEZ, G., (1995). On prime Galois coverings of Riemann sphere, *Ann. Mat. Pura Appl.*, **168**, IV, 1–15. DOI: 10.1007/BF01759251

[18] GROMADZKI, G.; WEAVER, A. AND WOOTTON, A., On gonality of Riemann surfaces, *Geom. Dedicata*, *to appear*. DOI: 10.1007/s10711-010-9459-x

[19] GROMADZKI, G., (2008). *On conjugacy of p-gonality automorphisms of Riemann surfaces*, *Rev. Mat. Complut.*, **21**, 1, 83–87.

[20] MACBEATH, A. M., (1973). Action of automorphisms of a compact Riemann surface on the first homology group, *Bull. London Math. Soc.*, **5**, 1, 103–108. DOI: 10.1112/blms/5.1.103

[21] TYSZKOWSKA, E., (2005). Topological classification of conformal actions on elliptic-hyperelliptic Riemann surfaces, *J. Algebra*, **288**, 2, 345–363. DOI: 10.1016/j.jalgebra.2005.03.024

[22] TYSZKOWSKA, E., (2008). Topological classification of conformal actions on 2-hyperelliptic Riemann surfaces, *Bull. Inst. Math. Acad. Sinica*, **33**, 4,345–368.

[23] TYSZKOWSKA, E., (2005). On p-hyperelliptic involutions of Riemann surfaces, *Beiträge zur Algebra und Geometrie Contributions to Algebra and Geometry*, **46**, 2, 581–586.

[24] TYSZKOWSKA, E. AND WEAVER, A., (2008). Exceptional points in the eliptic-hyperelliptic locus, *J. Pure Appl. Algebra*, **212**, 1415–1426.

[25] WEAVER, A., (2004). *Hyperelliptic surfaces and their moduli*, *Geom. Dedicata*, **103**, 69–87.

Ewa Tyszkowska
Institute of Mathematics,
Gdańsk University,
Wita Stwosza 57,
80-952 Gdańsk,
Poland