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Mathematical Methods in Modern Risk Measurement: A Survey*

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Abstract. In the last ten years we have been facing the development on new approaches in Risk Measurement. The Coherent, Expectation Bounded, Convex, Consistent, etc. Risk Measures have been introduced and deeply studied, but there are many open problems that will have to be addressed in forthcoming research. The present paper attempts to summarize the achieved findings and the "State of the Art", as well as their relationships with other Mathematical Fields, with special focus on other usual topics of Mathematical Finance.

Una panorámica sobre los métodos matemáticos de la moderna medición de riesgos

Resumen. En los últimos diez años hemos asistido al desarrollo de nuevos enfoques en Medición de Riesgos. Las Medidas de Riesgo Coherentes, Acotadas por la Media, Convexas, Consistentes, etc., han sido introducidas y profundamente estudiadas, aunque siguen abiertos numerosos problemas que tendrán que ser abordados en investigaciones futuras. El presente artículo sintetiza los logros alcanzados y "El Estado Actual de la Cuestión", así como las relaciones con otros campos de la Matemática, con atención especial a los temas clásicos de la Matemática Financiera.

1 Introduction

Perhaps the most important issues in Mathematical Finance are the Theory of Asset Pricing, the Theory of Portfolio Choice and the Risk Management Theory. Obviously, the three topics are closely related and those findings that are significant in a field have also interesting influence in the remaining ones. The risk measurement is a critical point affecting all the major topics (pricing, hedging, portfolio optimization, risk management, etc.). However, there is no a general method to measure the degree of risk of every financial strategy. On the contrary, there are alternative approaches and the use of a concrete one mainly depends on the specific problem we have to deal with. The risk study and measurement is also crucial in Actuarial Sciences, where the Classical Risk Theory will always play a central role.

The recent growing development of the Financial and Actuarial sectors has provoked a new point of view when measuring risk levels. The new approach must consider the necessities of regulators, who must

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provide rules guaranteeing the stability of the system, supervisors, who must control that the industrial activities respect the legal framework, and public or private companies, that must manage the wealth of their customers. In the European Union the set of rules that the industry must respect are mainly contained in Basle II (finance) and Solvency II (insurance). They provide the way that any corporation must follow in order to compute its "capital reserves", i.e., additional capital that will be devoted to overcome those periods characterized by losses of the economic activity. The size of the appropriate reserve may be considered as the risk level associated with the firm (or its activity). From a mathematical perspective this reserve will be understood as a real-valued function on a $L^p(\Omega, \mathcal{F}, \mu)$ space, where $p \in [1, \infty]$, and $(\Omega, \mathcal{F}, \mu)$ is an arbitrary probability space. The study of these functions, that we will call modern risk measures, is the focus of the Modern Risk Measurement.

This survey tries to summarize the most important mathematical findings concerning the modern risk measures. For the sake of brevity we will not be exhaustive, but we will try to provide the reader with a general view about the current "State of the Art". The modern measures of risk will merit our attention for various reasons: Firstly, as said above, they are related to the current legal framework that affects the financial companies. Secondly, they are far less known than more classical risk measures. Thirdly, they present many open and interesting problems that may be addressed by young researchers interested in these issues, and fourthly, they may apply in any financial or insurance problem as well as for every kind of risk (market risk, credit risk or operational risk).

As already said there are several classical approaches about the risk measurement, and they are far of being equivalent. Then we will also present a synthesis about the History of the (classical) Risk Measurement, since this information may assist to understand the necessity of the new approach. For the sake of simplicity we will concentrate our historical synopsis on the financial case, since presenting the actuarial perspective would significantly enlarge the article.

Throughout the article we will attempt to draw on the economic intuition and interpretation. Of course, many mathematical details will not be addressed. They may be found in the cited references, among many other books and articles. Furthermore, the interested reader may also consult the web `http://www.gloriamundi.org`, where a vast number of documents on risk measurement are available. For example, many here cited references can be downloaded from this webpage.

The first, second, third and fourth sections of the paper will yield a general view about the History of the Classical Risk Measurement in different financial markets (stock markets, fixed income markets and derivative markets). Sections 5, 6, 7, 8 and 9 will deal with the modern viewpoint, and the last section concludes the article.

2 Classical methods in risk measurement: Dispersion measures

Since the nineteenth century many authors significantly contributed the development in Mathematical Finance. However, in order to simplify this exposition, we will start our synopsis by summarizing the ideas of Harry **Markowitz**, 1990 **Nobel Laureate in Economics** due to his "Portfolio Theory" and his analysis if the Risk Measurement, amongst many other innovations. Indeed, Markowitz proved that normally and log-normally distributed share returns make the return variance compatible with the usual utility functions of Financial Economics (increasing and concave functions). Accordingly, he proposed his well-known method in Portfolio Optimization, that consists in simultaneously maximizing the expected portfolio return and minimizing the standard deviation of this return. Technical details may be found in [38] and [39].

Though he focused on the standard deviation as a risk measure, it is not difficult to extend his ideas so as to show that other dispersion measures (the absolute deviation, for instance) also respect the classical utility functions and, consequently, they can also be used in order to measure risk levels, despite the corresponding portfolio optimization problem may become much more complex.

Markowitz was born in Chicago in 1927 and he devoted his research to the applications of Mathematics in practical problems. For instance, the sentences below were written by himself:

When it was time to choose a topic for my dissertation, a chance conversation suggested the possibility of applying mathematical methods to the stock market. I asked Professor Marschak what he thought. He thought it reasonable, and explained that Alfred Cowles himself had been interested in such applications. He sent me to Professor Marshall Ketchum who provided a reading list as a guide to the financial theory and practice of the day.

I left the University of Chicago and joined the RAND Corporation in 1952. Shortly thereafter, George Dantzig joined RAND. While I did not work on portfolio theory at RAND, the optimization techniques I learned from George (beyond his basic simplex algorithm which I had read on my own) are clearly reflected in my subsequent work on the fast computation of mean-variance frontiers (Markowitz (1956) and Appendix A of Markowitz (1959)). My 1959 book was principally written at the Cowles Foundation at Yale during the academic year 1955-56, on leave from the RAND Corporation, at the invitation of James Tobin. It is not clear that Markowitz (1959) would ever have been written if it were not for Tobin's invitation.

My article on "Portfolio Selection" appeared in 1952. In the 38 years since then, I have worked with many people on many topics. The focus has always been on the application of mathematical or computer techniques to practical problems, particularly problems of business decisions under uncertainty. Sometimes we applied existing techniques; other times we developed new techniques. Some of these techniques have been more "successful" than others, success being measured here by acceptance in practice.

Another important contribution in Portfolio Theory was simultaneously presented by Roy (see [51]). He also maximized the expected return of a portfolio of stocks but the risk level was measured by the probability of losing money. Nevertheless, since this probability is bounded from above by using variances and the Tchebycheff inequality, in practical situations the variance becomes the risk measure once more.

William **Sharpe** (1964) —[55]— published the Capital Asset Pricing Model (*CAPM*), probably one of the most important Equilibrium Models in Financial Economics. Parallel work was also performed by Tobin ([56]), Treynor ([57]), Lintner ([36]) and others. The *CAPM* extended the Harry Markowitz's Portfolio Theory so as to introduce the notions of systematic (justified by "the market" or the "Market Portfolio") and specific (non justified) risk. Once again risk levels were measured with variances. For his work on $CAPM$, Sharpe shared the 1990 **Nobel Prize in Economics** with Harry Markowitz and Merton Miller.

Stephen A. **Ross** is famous for the invention of the Arbitrage Pricing Theory (*APT*), a key innovation in Financial Economics that partially extends *CAPM*, in the sense that the role of "the market" is plaid by other economic factors (inflation, interest rates, exchange rates, oil prices, etc.). He was awarded the **Graham & Dodd Prize** for his distinguished research on such wide-ranging work in mathematical finance theory as the theory of agency, the binomial model of option pricing, and the Cox-Ingersoll-Ross term structure model of interest rates.

More authors dealt with the variance and generalized the contributions above by considering strictly weaker assumptions about the asset returns behavior, the agents utility function or other involved economic properties. The Stochastic Discount Factor (*SDF*) is a key notion developed from the Riesz Representation Theorem (the isomorphism between a Hilbert Space and its Dual Space). The *SDF* leads to the Market Portfolio of Sharpe and the *CAPM* and *APT* major formulas. The existence of the *SDF* is equivalent to the fulfillment of the Law of One Price, necessary condition to guarantee the absence of arbitrage,[1] and it also connects with the theory by "Arrow-Debreu" about State Prices. In particular, the *SDF* generates "Pricing Rules", i.e., it provides linear functions enabling us to price new securities introduced in the market, such us derivatives.[2] The interested reader may consult, for instance, Chamberlain and Rothschild ([18]), Balbás, Mirás and Muñoz ([9]) and Cochrane ([20]).

[1] In Financial Economics and Mathematical Finance an arbitrage is an investment strategy that never leads to capital losses and may provide infinite returns under favorable scenarios. The existence of arbitrage is not compatible with the existence of equilibrium, which implies that the arbitrage absence is usually imposed in theoretical approaches.

[2] A derivative security is an asset whose value depends on another asset called underlying security. Call and put options, as well as forward and future contracts, are the most popular derivatives, though there are more examples.

The ideas above were more recently addressed in models that incorporate transaction costs, portfolio constraints and/or other market imperfections or frictions. Under the new assumptions authors still deal with the variance (or other dispersion) as a risk measure and they attempt to find *SDF*, efficient strategies, the Market Portfolio, Pricing Rules, etc. There are many open questions in this field since problems become far more complex from a mathematical point of view. For instance, new separation theorems have been stated, extending those that are more classical in Functional Analysis. Important papers are, among others, He and Modest ([31]), Luttmer ([37]) and Jouini and Kallal ([33]).

Another question recently generating a growing interest is the compatibility between a Dispersion Measure and the "Second Order Stochastic Dominance" (*SOSD*).[3] As said above, Markowitz justified the variance by showing that it is compatible with usual utilities, but he only dealt with normal and log-normal distributions. Empirical studies are indicating that the presence of "asymmetries", "heavy tails" and "extreme values" is becoming more and more evident. In a Non-Gaussian world the arguments of Markowitz do not work any more, and the variance is not an appropriate risk measure. Indeed, it is not compatible with the *SOSD*, and minimizing the standard deviation one could find a portfolio that does not maximize any rational utility function. Recent research has shown that this caveat may be overcome by using alternative dispersion measures, such as the absolute deviation and semideviation, the standard semideviation etc. See Ogryczak and Ruszczynski ([41, 42, 43]) for further details.

The introduction of new dispersion measures is making portfolio choice problems much more complicated in practice. Another line of research addresses this topic. For example, Konno Akishino and Yamamoto ([35]) incorporate frictions and fat tails and minimize the absolute deviation, and a more sophisticated analysis may be found in Jouini and Kallal ([33]).

3 Classical methods: Sensitivities for fixed income securities

Seventy years ago Macaulay introduced the notion of "Duration" of fixed income securities or portfolios. The idea is quite simple from a mathematical viewpoint. Indeed, the market price of every bond depends on the interest rate level. If this level increases (decreases) then the price decreases (increases) and the "slope" of this negative relationship indicates how much money the bondholder can lose due to the possible interest rate growth.[4] The same idea applies if we deal with a general strategy composed of fixed income securities, although the negative derivative does not necessarily hold if there are short positions (sold securities) involved. This is the reason why Macaulay considered that the derivative of the portfolio price with respect to the interest rate level may be understood as a risk measure. He also showed that this derivative is closely related to "his duration", a particular date that depends on the strategy and the interest rate level.

The ideas of Macaulay were later formalized in Fisher and Weil ([25]), where the authors showed that a fixed income portfolio is "Immunized" with respect to the interest rate risk if and only if the duration of assets equals the duration of liabilities. In other words, the portfolio is immunized if and only if the derivative of the global portfolio price with regard to the interest rate vanishes.

However, in practice the interest rate is not constant and critically depends on the considered period, arising the so called Term Structure of Interest Rates (*TSIR*). Furthermore, interest rate shifts are non-flat, in the sense that they are also functions of time. This fact has motivated further extensions of the Fisher and Weil approach. Some examples are Chambers, Carleton and McEnally ([19]), Elton, Gruber and Michaely ([24]), Bowden ([16]), Paroush and Prisman ([44]), Barber and Copper ([12]) or Agca ([1]). All of them deal with a non-flat *TSIR* and non-flat shifts, and they measure the portfolio risk by using partial derivatives or differentials, that must vanish to prevent the existence of risk.

Fong and Vasicek ([28]) seems to be the first paper using a dispersion measure, called M-squared and closely related to the variance, that may measure the risk level with respect to the *TSIR* modifications.

[3] See Yaari ([60]) for details about this notion.

[4] Notice that the interest rate growth will never be "too large" in practice (in a short period of time), so the function indicating the bond price may be identified with its first order approximation.

They also showed that M-squared is given by a mathematical derivative as well. Therefore, they established some kind of relationship between a risk measure given by a derivative and a risk measure given by a dispersion. As far as we know this is the first paper addressing this topic. The ideas of Fong and Vasicek were empirically tested in many papers (for instance, Bierwag, Fooladi and Roberts, [14]), and their theoretical contribution was significantly extended by Balbás and Ibáñez ([7]) and by Balbás, Ibáñez and López ([8]), where the authors studied many properties connecting dispersions and derivatives for fixed income strategies, as well as between dispersions and possible capital losses.

4 Sensitivities of derivative portfolios: "The Greeks"

Professor Robert C. **Merton**, Harvard University, Cambridge, USA and Professor Myron S. **Scholes**, Stanford University, Stanford, USA, 1997 **Nobel Laureate in Economics**.

Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer ***Black****, developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society.*

As indicated in the lines above, Merton and Scholes were awarded 1997 Nobel Prize for their work in collaboration with Black, who had already died in 1997.

The three authors proposed a continuous time approach to explain a stock price stochastic behavior, and then they drew on the Ito's Lemma in order to reach a second order partial differential equation that must be satisfied by the price of any replicable derivative (see Black and Scholes [15]). By solving the equation, along with terminal conditions that depend on the derivative to be analyzed, derivatives can be priced ad hedged.

The Girsanov-Martin-Cameron theorem provides an alternative way since it gives the risk-neutral pricing rule, i.e., it allows as to modify the real probability so as to obtain a new probability measure, called risk-neutral, and such that it is equivalent to the initial one and the discounted market price become a martingale under the risk-neutral measure. Thus, every derivative can be priced if we know the (risk-neutral) distribution of its terminal price (or pay-off). Indeed, the pricing rule consists in computing the expectation of the terminal price present value.

According to the Black and Scholes conclusions, the risk level generated by any derivative security may be neutralized by trading δ units of the underlying asset, δ being the partial derivative (sensitivity) of the derivative security price with respect to the underlying asset price. Thus, once again risk levels can be measured by sensitivities. Since δ is dynamic and random (or stochastic), to hedge a long/short position in a derivative agents must continuously rebalance the quantity of the underlying asset in the portfolio. But practitioners cannot trade in a continuous time setting, which may lead to new sources of risk. So, the usual Γ, the sensitivity of δ with respect to the underlying asset price, will also play an important role when hedging derivative portfolios. Actually, if Γ is not close to zero then the value of δ is not stable, and the portfolio must be rebalanced much more often.

In order to prevent discrepancies between the real market behavior and the model dynamic assumptions, some more sensitivities must be "under control". So, a perfect hedging of derivative portfolios implies that the famous "Greeks" must (if possible) vanish. The Greeks are therefore very important risk measures. They are:

Delta (δ). Sensitivity (or partial derivative) of the portfolio price with respect to the underlying asset price.

Gamma (Γ). Sensitivity of Delta with respect to the underlying asset price.

Theta (θ). Sensitivity of the portfolio price with respect to time.

Ro (ρ). Sensitivity of the portfolio price with respect to the riskless interest rate.

Vega (υ). Sensitivity of the portfolio price with respect to the underlying asset volatility.

More recently some more Greeks have been introduced, though the basic ideas remain the same as above.

During many years these properties have been extended to more complex stochastic frameworks (stochastic volatility, Heston model, jumps in the volatility, interest rates stochastic models, currencies, volatility-linked derivatives, etc.). This has motivated the development of the so called "Asset Pricing Theory", a very important topic in Mathematical Finance. Asset Pricing Theory is closely related to Risk Management in both complete and incomplete markets,[5] though things are far more complex in incomplete models. The interested reader may consult, for instance, Back ([11]). "Abstract pricing models" have also deserved the attention of many researchers. They do not impose any special assumption about the price behavior. The price process is just a stochastic process, and, under suitable conditions, the absence of arbitrage is characterized by the existence of an equivalent martingale measure.[6] Moreover, the market is complete if and only if the martingale measure is unique.[7] The existence of martingale measures has also been addressed under the existence of frictions (see Jouini and Kallal ([33]) or Schachermayer ([53]), amongst many others). Further extensions of the Fundamental Theorems of Asset Pricing go on being studied in both perfect or imperfect markets.

5 Recent measures: Capital requirements and risk functions

As illustrated above, the risk measurement critically depends on the securities we are dealing with, making it rather complicated to introduce a global measure applying for different financial instruments and kinds of risk.

On the other hand, along the twentieth century many financial crises provoked the investors lack of confidence in financial institutions. It made rather convenient the introduction of margins and financial requirements by regulators.

On June 26th, 1974, German Regulators forced Bank Herstatt into liquidation due to its level of losses. This case and others motivated the group G-10 (Belgium, Canada, France, Germany, Italy, Japan, Netherlands, Sweden Switzerland UK and US) to create the Basle Committee on Banking Supervision, which provided a minimal set of capital requirements in 1988.

In 1993 the European Union also integrated several systems of Capital Requirements, involving both, Banks and other Financial Institutions. The system of provided rules was called Basle I. Later, the process has been extended in many directions all around the world.

Two years ago the European Commission provided Basle II, a new complex set of rules that Banks and other Financial Institutions must respect in order to compute their margins and Capital Requirements. Basle II is much more sophisticated that Basle I because it incorporates the level of complexity of modern capital markets, and it also allows the involved institution to develope particular risk measurement procedures if they can be adequately justified with theoretical and practical arguments. The supervisor (the Bank of Spain or *Comisión Nacional del Mercado de Valores*, in the Spanish case) must decide if the created control system is correct. This new legal framework will cause competition between different financial institutions, that must look for "cheap and efficient" ways to measure and control the degree of risk.

The European Commission will publish Solvency II in one or two years, quite similar to Basle II but specially affecting the Insurance Firms.

[5] A pricing model is said to be complete if every terminal price (or pay-off) may be replicated by a self-financing portfolio, i.e., given a pay-off f paid at a future date T, there exists a stochastic portfolio that will not generate any pay-off before T and will pay f at T.

[6] This is the First Fundamental Theorem of Asset Pricing. See Harrison and Kreps ([30]) for a simple version or Dalang, Morton and Willinger ([22]) and Jacod and Shiryaev ([32]) for more complex versions under far weaker assumptions.

[7] Second Fundamental Theorem of Asset Pricing. See Harrison and Kreps ([30]), Dalang, Morton and Willinger ([22]) or Jacod and Shiryaev ([32]).

Value at Risk

From an intuitive point of view the "Value at Risk" (*VaR*) is just a percentile. During the late 1980's JP Morgan developed a firm-wide *VaR* system, in order to integrate many types of risks and portfolios as well as many capital requirements in a "single number". They assumed normal distributions in order to estimate.

Actually, a correct definition of *VaR* that applies for every random variable y on the probability space $(\Omega, \mathcal{F}, \mu)$ is

$$VaR_{\mu_0}(y) = -\inf\{\alpha \in \mathbb{R} : \mu(y \leq \alpha) > \mu_0\},$$

$0 < \mu_0 < 1$ being the confidence level.[8] It is assumed that $VaR_{\mu_0}(y)$ yields the capital reserve that the manager must add in order to prevent losses. This margin or reserve will be effective unless we face the $100\mu_0\%$ worst scenarios.

It is easy to see that *VaR* and the variance are closely related if we are dealing with normal distributions. Indeed, in this case we have

$$VaR_{\mu_0}(y) = -m - \sigma\Phi^{-1}(\mu_0) \tag{1}$$

m and σ denoting the expected value and the standard deviation of y, and

$$\Phi(\mu_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\mu_0} e^{-\frac{s^2}{2}}\, ds$$

being the cumulative standard normal distribution. Thus, if (as usual) the confidence level is less that 50%, then in a Gaussian world the maximization of the expected return and the simultaneous minimization of VaR_{μ_0} is almost equivalent to the Markowitz model.

6 Coherent measures of risk.

VaR_{μ_0} is a risk measure that may present significant drawbacks in a non-Gaussian world, as we will indicate. The coherent measures of risk were introduced in Artzner, Delbaen, Eber and Heath ([2]). These authors pointed out that the initial capital requirement associated with a final random pay-off y may be measured by several functionals on the space of random variables y with finite expectation. Though in their first approach they considered probability spaces with a finite number of events, consider a general probability space $(\Omega, \mathcal{F}, \mu)$, fix $p \in [1, \infty]$, and a coherent risk measure is a function

$$R : L^p(\Omega, \mathcal{F}, \mu) \longmapsto \mathbb{R}$$

such that:

- $R(y + y_0) = R(y) - y_0$ for every $y \in L^p$ and every $y_0 \in \mathbb{R}$ (translation invariance).[9]
- $R(y + z) \leq R(y) + R(z)$ for every $y, z \in L^p$ (subadditivity).[10]
- $R(y) \geq R(z)$ for every $y, z \in L^p$ with $y \leq z$ almost everywhere (monotonicity).[11]
- $R(\lambda y) = \lambda R(y)$ for every $y \in L^p$ and every $\lambda \in \mathbb{R}$ with $\lambda > 0$ (homogeneity).

Moreover, they also considered the axiom below, although it is not imposed to general coherent risk measures:

[8] We are assuming that y represents a future price or pay-off, i.e., we are taking a financial perspective. If y represents losses (actuarial perspective) then the correct definition becomes

$$VaR_{\mu_0}(y) = \sup\{\alpha \in \mathbb{R} : \mu(y \geq \alpha) > \mu_0\}.$$

[9] We write $y_0 \in \mathbb{R}$ because we identify every real number with the obvious constant function of $L^p(\Omega, \mathcal{F}, \mu)$.

[10] Subbaditivity is imposed because portfolio choice problems should lead to very diversified portfolios.

[11] An agent receiving z will be richer than an agent receiving y, so the risk level of the first investor must be lower.

- $R(y) < 0$ if $\mu(y \geq 0) = 1$ and $\mu(y > 0) > 0$ (relevance).

The authors also defined "the acceptance set" associated with the measure R. It is given by

$$\mathcal{A}_R = \{y \in L^p : R(y) \leq 0\}.$$

Since the translation invariance implies that $R(y + R(y)) = 0$, once the trader adds the quantity (cash) $R(y)$ to that investment paying y the global portfolio is acceptable.

All the properties above seem to be natural and present nice economic interpretations. Moreover, the authors showed that *VaR* is not a coherent measure because it is not subadditive. It was illustrated with a simple counterexample incorporating two digital options with the same underlying asset, maturity and expiration date. Hence, using *VaR* in some portfolio choice problems might lead to non diversified strategies.

Consequently, other authors proposed different kind of risk measures (also called risk functions) non necessarily subadditive. For instance, the paper by Goovaerts, Kaas, Dhaene and Tang ([29]) showed that even superadditivity may be convenient for some kind of insurance problems. Anyway, Artzner, Delbaen, Eber and Heath yielded several interesting examples of quantile-linked coherent measures. Some of them are the Tail Conditional Expectation (*TCE*) and The Worst Conditional Expectation (*WCE*), that are defined in their paper.[12]

It is worthwhile to mention that *TCE* is often called Conditional Value at Risk (*CVaR*) or Expected Shortfall (*ES*). For instance, the articles by Rockafeller and Uryasev (2000) and (2002) provides a nice comparison between the properties of *VaR* and *CVaR*. For normal distributions there is a simple relationship between *VaR*, *CVaR* and standard deviations. Indeed, with the notations already introduced in (1), Expressions

$$CVaR_{\mu_0}(y) = -m + \frac{\sigma}{\mu_0\sqrt{2\pi}} e^{-\frac{\left(\Phi^{-1}(\mu_0)\right)^2}{2}}$$

and

$$CVaR_{\mu_0}(y) = -m + \frac{\sigma}{\mu_0\sqrt{2\pi}} e^{-\frac{\left(\frac{VaR_{\mu_0}(y)+m}{\sigma}\right)^2}{2}}$$

hold. Once again, in a Gaussian world the minimization of *CVaR* is almost equivalent to the Markowitz model.

As we will see, risk measures may be quite useful in portfolio choice problems. In such a case it is interesting to remark that *VaR* may underestimate the risk level. Indeed, according to the Artzner , Delbaen, Eber and Heath results, under appropriate assumptions (a finite probability space, for instance) one has

$$VaR_{\mu_0}(y) = \inf\{R(y) : R \text{ coherent and } R \geq VaR_{\mu_0}\}.$$

7 Representation Theorems

Artzner , Delbaen, Eber and Heath gave very interesting representation theorems for their risk measures. Mainly, since coherent risk measures are convex functions, they represented the measure by its subgradient, given by a family of finitely-additive $\mu-$continuous measures on $\mathcal{F}$. Their ideas have been extended in several directions and, for instance, Föllmer and Schied ([27]) used weaker axioms and introduced the Convex Risk Measures, along with various representation theorems. For the sake of brevity we will not present the representation theorems above with precision, and we will concentrate on the approach by Rockafellar, Uryasev and Zabarankin ([48]). They defined an Expectation Bounded Risk Measure as a real valued function on L^p that satisfies the translation invariance, homogeneity, subadditivity and, finally,

[12] Basically, these measures almost yield the expected losses if one faces the $100\mu_0\%$ worst scenarios, i.e., if one loses more than VaR_{μ_0}. Whence, these measures are conditional mathematical expectations of y.

$R(y) > -\int_{\Omega} y\,d\mu$, for every $y \in L^p$ non constant (out of null sets) (mean dominating).

They do not draw on finitely-additive $\mu-$continuous measures on $\mathcal{F}$ in order to represent R. They take $p < \infty$ and use the duality (L^p, L^q) so as to state that for every Expectation Bounded Risk Measure R the equality

$$R(y) = \max\left\{-\int_{\Omega} yz\,d\mu : z \in \Delta_R\right\} \tag{2}$$

holds for every $y \in L^p$, $\Delta_R \subset L^q$ being the convex and $\sigma(L^q, L^p)$-compact set given by

$$\Delta_R = \left\{z \in L^q : -\int_{\Omega} \tilde{y}z\,d\mu \leq R(\tilde{y}) \text{ for every } \tilde{y} \in L^p\right\}$$

The authors also provided the subgradient Δ_R of many expectation bounded risk measures. For instance, if $R = CVaR_{\mu_0}$ then $p = 1$, $q = \infty$, and

$$\Delta_{CVaR_{\mu_0}} = \left\{z \in L^{\infty} : 0 \leq z \leq \frac{1}{\mu_0} \; and \; \int_{\Omega} z\,d\mu = 1\right\}.$$

Finally, Rockafellar, Uryasev and Zabarankin showed that R is coherent if and only if Δ_R is included in positive cone of L^q.

Distortion functions

Another way to introduce risk measures (non necessarily coherent or expectation bounded) is the use of "distorting functions". So, according to the Wang ([58]) approach, if

$$g : [0,1] \rightarrow [0,1]$$

is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$, and $p \in [1, \infty]$, then many risk functions $R_g : L^p(\Omega, \mathcal{F}, \mu) \longmapsto \mathbb{R}$ may be given by the heuristic Stieltjes integral

$$R_g(y) = \int_0^1 VaR_t(y)\,dg(t). \tag{3}$$

This new method yields much more practical risk functions. Examples are the measure of Wang, that appears if we fix $p = 2$ and $a > 0$ and take

$$g(t) = \Phi\left(\Phi^{-1}(t + a)\right) \tag{4}$$

and the Dual Power Transform ($p = 1$), generated by

$$g(t) = 1 - (1 - t)^a$$

where $a > 1$ is a fixed constant.[13] Furthermore, many other measures may be represented by a distortion function. For instance

$$g(t) = \begin{cases} 0, & t < \mu_0 \\ 1, & t \geq \mu_0 \end{cases}$$

applies for VaR_{μ_0} and

$$g(t) = \begin{cases} t/\mu_0, & t < \mu_0 \\ 1 & t \geq \mu_0 \end{cases}$$

for $CVaR_{\mu_0}$.

Wang showed that the corresponding risk function R_g in (3) may respect some properties related to the *SOSD* and utility functions. Finally, Wang proved that the use of the risk function R_g given by (4) may apply for pricing financial instruments. In particular, he showed that the classical *CAPM* and the Black and Scholes model may be included into the pricing rules generated by his risk function.

[13]Let us remark that from an economic viewpoint it is possible to prove in both examples that the degree of risk aversion increases if so does a.

Pricing, hedging and portfolio choice problems

There is another line of research that provides new methods for hedging (and therefore pricing) in incomplete markets. In such a case a perfect hedging is not always available, and "a partial hedging" may be addressed by minimizing the risk of final losses. The risk level is given by a general risk function and the optimization problem consists in finding the appropriate (or optimal) dynamic self-financing portfolio. The problem may be quite complex from a mathematical viewpoint, but under some general assumptions it may be simplified, as pointed out in Föllmer and Leukert ([26]). Further extensions of this work were recently given in Nakano ([40]) and Schied ([54]).

Besides, Portfolio Choice Problems may be also addressed by using general risk functions. As already said, asymmetries, fat tails and capital requirements may make it convenient to abandon the variance and to use alternative risk functions, since the compatibility with the *SOSD* must be retrieved and the optimal portfolio must be related to the reserves to be added by the manager. The optimization of a general risk function may be complex since, for instance, coherent and expectation bounded risk measures are non-differentiable, and classical optimization methods may fail. Recent contributions to this issue have been given in Benati ([13]), where the author optimizes the *WCE*, or in Rockafellar, Uryasev and Zabarankin ([49]) and Ruszczynski and Shapiro ([52]), where more general optimization issues are analyzed. Balbás, Balbás and Mayoral ([5]) draw on Expression (2) so as to introduce an alternative Linear Programming Problem in General Banach Spaces that is equivalent to a (non linear and non differentiable) standard portfolio choice problem. So, if R is a risk function, $Y \subset L^p$ and we face the optimization problem

$$\begin{cases} \min R(y) \\ \quad y \in Y \end{cases}$$

then, by applying (2) we can consider the equivalent problem

$$\begin{cases} \min & \theta \\ & \theta + \int_\Omega yz \, d\mu \geq 0, \quad \forall z \in \Delta_R \\ & \theta \in \mathbb{R}, \quad y \in Y \end{cases} \tag{5}$$

whose decision variable is $(\theta, y) \in \mathbb{R} \times L^p$. Regardless of the properties of R, notice that Problem (5) becomes differentiable under "a reasonable" set of constraints Y, and the first constraint is $\mathcal{C}(\Delta_R)$-valued, $\mathcal{C}(\Delta_R)$ being the Banach space of continuous and $\mathbb{R}$-valued functions on the compact space Δ_R. Thus, Lagrange and Karush-Kuhn-Tucker like conditions may apply, with multipliers belonging to $\mathcal{M}(\Delta_R)$, space of Radon measures on the compact Δ_R and dual of $\mathcal{C}(\Delta_R)$. Moreover, (5) will be often linear, in which case Balbás, Balbás and Mayoral solve practical examples by adapting a simplex-like algorithm for infinite-dimensional problems inspired in Balbás and Romera ([10]), where a hedging problem for the interest rate risk is studied.

8 Vector and dynamic risk measures.

In a recent paper Jouini Meddeb and Touci ([34]) justified the use of vector risk functions. The authors extended the ideas of Artzner, Delbaen, Eber and Heath and defined their "Coherent Vector Risk Measures". They are set valued functions

$$R : L^\infty(\Omega, \mathcal{F}, \mu, \mathbb{R}^n) \longmapsto \mathbb{R}^{\tilde{n}},$$

$n, \tilde{n} \in \mathbb{N}$, $n \leq \tilde{n}$, such that:

- $R(y)$ is closed, contains zero and is not $\mathbb{R}^{\tilde{n}}$.
- $R(y + y_0) = R(y) - (y_0, 1, ..., 1)$ if $y_0 \in L^\infty(\Omega, \mathcal{F}, \mu, \mathbb{R}^n)$ is constant (almost everywhere).

- $R(y+z) \subset R(y) + R(z)$, $y, z \in L^{\infty}(\Omega, \mathcal{F}, \mu, \mathbb{R}^n)$.
- $R(\lambda y) = \lambda R(y)$ if $y \in L^{\infty}(\Omega, \mathcal{F}, \mu, \mathbb{R}^n)$ and $\lambda > 0$.

The authors showed that this is a genuine extension of the scalar case and justified the definition by considering several kinds of risk. They provide a Representation Theorem by using finitely additive vector measures on $\mathcal{F}$. To prevent the use of some dual spaces they draw on the so called "Fatou Property". The paper ends by addressing which the authors called "Coherent Aggregation of Risk".

An alternative approach is given in Balbás and Guerra ([6]). In this case the authors leave the set-valued functions and consider a Vector Risk Measure as a function

$$R : L^p(\Omega, \mathcal{F}, \mu, \mathsf{Y}) \longmapsto \mathsf{Z},$$

$p \in [1, \infty]$, and Y and Z being general Banach Lattices. They introduce the concepts of Deviation, Coherent Risk Function and Expectation Bounded Risk Function, and for all of them they yield representation theorems that extend (2). The paper ends by giving several practical examples and general methods to optimize vector risk functions.

There are some more papers dealing with closely related issues. For instance, Burget and Rüschendorf ([17]) extend the concept of Consistent Risk Measure of Goovaerts, Kaas, Dhaene and Tang ([29]), in the sense that they work with real-valued functions such that the argument belongs to $L^p(\Omega, \mathcal{F}, \mu, \mathbb{R}^n)$. In the same line Detlefsen and Scandolo ([23]) consider real valued functions whose arguments are stochastic processes satisfying some required conditions. These authors point out the existence of relationships between vector and dynamic risk measures.

Dynamic risk functions

There is no a consensus about how to define a dynamic risk measure. Since real capital markets are dynamic, it seems to be clear that we need a dynamic approach to measure risk levels, but the problem is how we can do that. Besides, regulators, supervisors and managers need a stable number to establish the risk level and the capital requirements of a given corporation, so some researchers think that the use of dynamic measures might not be so convenient.

An interesting approach was given by Cvitanic and Karatzas ([21]), who considered "static" risk measures, but they were computed by taking into account that the investor may rebalance her/his portfolio, i.e., he/she lives in a dynamic world. Since then this method has been frequently applied, and it is usual to deal with a dynamic *VaR* or *CVaR*, amongst other risk functions. They are the same measures as we defined above, but they do not apply to the final pay-off y of the present portfolio. Indeed, y is modified to $\tilde{y}$, final pay-off of some alternative strategies generated by possible modifications of the initial portfolio. These modifications have to be constrained by adequate conditions.

Nevertheless, some authors have attempted to extend the risk measurement methods to a dynamic setting, in the sense that a risk measure must be a stochastic process that must reflect the possible evolution of the risk level in future dates.

More importantly, an alternative idea is to consider that the risk level is just a real number, but the argument is not a random variable but a stochastic process, i.e., we do not consider the present portfolio final price, but the whole stochastic portfolio that is not restricted except for the initial date (the present portfolio). In this line Artzner, Delbaen, Eber, Heath and Ku ([3] and ([4]) define the coherent dynamic risk measures in a discrete time framework, and provide representation theorems that extend their previous work. However, it seems to be quite difficult to extend their approach to a continuous time setting, since their representation theorem uses a tensor product of σ-algebras that seems to be hardly generalizable for non-countable families.

As already said, Detlefsen and Scandolo ([23]) also addressed possible ways to introduce dynamic measures. They are related to vector measures and, in some sense, also to the ideas of Artzner, Delbaen, Eber, Heath and Ku.

9 Integrating the approaches above

An open problem is the integration of the approaches above. There are partial results (the interested reader may consult the provided references and websites) concerning the relationships between utilities, stochastic dominance, sensitivities, dispersions and margin requirements. For instance, Rockafellar, Uryasev and Zabarankin ([48]) have introduced the axioms that a "Deviation Measure" must satisfy, and they have established the existence of a one to one correspondence between deviations and expectation bounded risk measures. For instance, there exists an expectation bounded risk measure related to the standard deviation, and the minimization of both functions is equivalent. Unfortunately, this expectation bounded risk measure is neither coherent nor compatible with the *SOSD*, so it is not so good to reflect capital margins if skewness and heavy tails are involved.

Several authors have been dealing with those properties guaranteeing that a risk function is compatible with the *SOSD*. Amongst many others, Wirch and Hardy ([59]) have stated that the risk function R_g of (3) is compatible with the *SOSD* if and only if g is strictly concave, in which case R_g is coherent too. Hence, the Dual Power Transform and the measure of Wang are compatible with the *SOSD*, but the *CVaR* fails to be. A complementary approach concerning the compatibility with the *SOSD* may be found in Pflug ([45]), where the topic is also related to the possible representation of the risk function to be studied.

Despite the partial results above, and many others, a general analysis concerning the integration of the presented approaches about the risk measurement has not been developed yet. At least, a much deeper study on the relationship between classical (dispersions and sensitivities) and modern (capital reserves) risk functions should be quite important, because the classical hedging methods and strategies still apply and will go on applying for their associated problems, owing to the good reflected practical performance. The problem is also important from the theoretical viewpoint, since it motivates deep mathematical developments and may provide new ways to address pricing and hedging issues.

10 Conclusions

The measurement of risk levels is a major topic in Mathematical Finance. It is related to major classical issues, like Hedging, Pricing and Portfolio Choice Problems, amongst others. In the past it has been addressed by drawing on different approaches, all of them reflecting a complex mathematical development. We have summarized some major findings, but for obvious reasons a lot of questions have not been addressed here. Many theoretical and practical problems are still open. Functional Analysis, Measure and Probability Theory, Differential Equations, Stochastic Calculus, Mathematical Programming and other mathematical fields play a crucial role, and they will go on playing a crucial role in future research.

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