RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **100** (1-2), 2006, pp. 75–100
Análisis Matemático / Mathematical Analysis
Artículo panorámico / Survey

The Projective Tensor Product II: The Radon-Nikodym Property

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Abstract. In this paper we discuss the problem of when the projective tensor product of two Banach spaces has the Radon-Nikodym property. We give a detailed exposition of the famous examples of Jean Bourgain and Gilles Pisier showing that there are Banach spaces X and Y such that each has the Radon-Nikodym property but for which their projective tensor product does not; this result depends on the classical theory of absolutely summing, integral and nuclear operators, as well as the famous Grothendieck inequality for its punch-line. In the last section of this paper we discuss many results of a positive character, due to Qingying Bu and various of his coauthors; in particular, we mention results of Bu, Diestel, Dowling and Oja to the effect that if one of the spaces has a boundedly complete FDD then the projective tensor product of two spaces with the RNP has it and a modification of a result of Bu and Pei-Kee Lin to the effect that if X is a Banachlattice with RNP and Y is any Banach space with RNP then their projective product has RNP.

El producto tensorial proyectivo II: La propiedad de Radon-Nikodym

Resumen. En este trabajo discutimos el problema de cuándo el producto tensorial proyectivo de dos espacios de Banach tiene la propiedad de Radon-Nikodym. Damos una exposición detallada de los famosos ejemplos de Bourgain y Pisier de dos espacios de Banach X e Y con la propiedad de Radon-Nikodym tales que su producto tensorial proyectivo no la tiene; este resultado depende de la teoría clásica de operadores absolutamente sumantes, integrales y nucleares, así como de la famosa desigualdad de Grothendieck como herramienta básica. En la última sección de este trabajo discutimos muchos resultados positivos, debidos a Qingying Bu y a varios de sus coautores; en particular, mencionamos resultados de Bu, Diestel, Dowling y Oja en la dirección de que si uno de los espacios tiene una FDD acotadamente completa, entonces el producto tensorial proyectivo de dos espacios con la RNP la tiene, y una modificación de un resultado de Bu y Pei-Kee Lin en el sentido de que si X es un retículo de Banach con la RNP e Y es cualquier espacio de Banach con la RNP entonces su producto tensorial proyectivo tiene la RNP.

Presentado por Vicente Montesinos Santalucía.
Recibido: 02/01/2006. *Aceptado*: 30/03/2006.
Palabras clave / Keywords: Projective tensor product; Radon-Nikodym property; absolutely summing operators; integral operators; nuclear operators.
Mathematics Subject Classifications: 46A32; 46B22.

1. Introduction

In this survey, we discuss when the projective tensor product of two Banach spaces has the Radon-Nikodym property. The topic is, admittedly, a narrow one; however, it is one area in which the projective tensor product exhibits strikingly regular stability results. Indeed, other than the remarkable set of examples of $\mathscr{L}_\infty$-spaces invented by Jean Bourgain and Gilles Pisier, the probablistic/measure theoretic basis upon which the Radon-Nikodym property is built seems to be a perfect fit for preservation in the projective mold.

The Bourgain-Pisier examples are very special indeed and so we spend considerable time and effort discussing them. Our presentation does not stray far from the original; its value, if any, is in the few added details provided, details that confounded us to some extent and, unprovided, might dissuade others from studying this amazing construct.

We open with a discussion of an abstract construction due to Sergei Kislyakov followed by some ramifications of that construct noted by Bourgain and Pisier.

In the next section, the basic ideas related to the Radon-Nikodym property make their entrance; this is followed by the main details of the construction. In the fourth section of this paper, we detail what all that goes before has to do with the *projective* tensor product. Here the theory of absolutely summing, integral and nuclear operators, accompanied by Grothendieck's ever-potent inequality, make their contributions.

In the last section, we discuss more recent results that renew the belief that the Bourgain-Pisier examples are indeed very special: our discussion centers around some work of Qingying Bu and various coauthors that establishes mainfold situations where spaces with the Radon-Nikodym property have a projective tensor product with the property as well.

Our terminology and notation is fairly standard. For sources, we call on the standard references on the subject of Banach spaces: [23] and [24], along with the still-fresh precursor [22] and wonderfully-informative overview [17]. For vector measures, we use [13] as our source.

2. Kislyakov Magic, As Practised By Bourgain And Pisier

The following construct was conjured up by S. Kislyakov and used by him, J. Bourgain and G. Pisier in remarkable ways; we hope to expose but a few of these.

Theorem 1 (Kislyakov) *Let S be a closed linear subspace of the Banach space B, E be a Banach space, $\eta \leq 1$ and $u : S \to E$ be a bounded linear operator with $\|u\| \leq \eta$.*

Then there exist a Banach space E_1, an isometric embedding $j : E \to E_1$ and an operator $\tilde{u} : B \to E_1$ such that $\|\tilde{u}\| \leq 1$,

$$\tilde{u}|_S = ju,$$

and E_1/E and B/S are isometrically isomorphic.

PROOF. Look at the ℓ^1 -direct sum $B \oplus_1 E$ of B and E; inside $B \oplus_1 E$ lies N

$$N = \big\{(s, -us) : s \in S\big\},$$

a closed linear subspace. Define E_1 by

$$E_1 = \big(B \oplus_1 E\big)/N.$$

Let $\pi : (B \oplus_1 E) \twoheadrightarrow E_1$ denote the natural quotient map and define

$$j : E \to E_1\ ,\ \tilde{u} : B \to E_1$$

as follows: for $e \in E, b \in B$

$$j(e) = \pi(0, e)\ \ \tilde{u}(b) = \pi(b, 0).$$

j is an isometry: For any $e \in E$

$$\begin{aligned}\|j(e)\| &= \|\pi(0,e)\| \\ &= \inf_{n \in N} \left\{ \|(0,e) - n\| \right\} \\ &\leq \left\|(0,e)\right\|_{B \oplus_1 E} = \|e\|.\end{aligned}$$

Further, for any $s \in S$

$$\begin{aligned}\left\|(0,e) - (-s, us)\right\| &= \left\|(-s, e - us)\right\| \\ &= \|-s\| + \|e - us\| \\ &\geq \|s\| + \|e\| - \|us\| \\ &\geq \|s\| + \|e\| - \|s\| \\ &= \|e\|\end{aligned}$$

so

$$\begin{aligned}\|j(e)\| &= \inf_{s \in S} \left\{ \|(0,e) - (-s, us)\| \right\} \\ &\geq \|e\|\end{aligned}$$

$\|\tilde{\mathbf{u}}\| \leq \mathbf{1}$: For any $b \in B$

$$\begin{aligned}\|\tilde{u}(b)\| &= \|\pi(b,0)\| \leq \|\pi\| \quad \|(b,0)\| \\ &\leq \|(b,0)\| = \|b\|.\end{aligned}$$

$\tilde{\mathbf{u}}|_{\mathbf{S}} = \mathbf{ju}$: For each $s \in S$, we have

$$\tilde{u}(s) = \pi(s,0) = (s,0) + N$$

and

$$ju(s) = \pi(0, u(s)) = (0, u(s)) + N$$

$$(s,0) - (0, u(s)) = (s, -u(s)) \in N$$

And so, as members of $(B \oplus_1 E)/N = E_1$,

$$\tilde{u}(s) = ju(s)$$

Now the fact that $\tilde{u}|_S = ju$ says that $\tilde{u}$ takes S into $j(E)$ and so $\tilde{u}$ ‘lifts’ to a linear operator $\tilde{U} : B/S \to E_1/j(E)$; the operator $\tilde{U}$ is given by

$$\tilde{U}(b + S) = \pi(b,0) + j(E) = \tilde{u}(b) + j(E).$$

From this it’s plain that $\tilde{U}$ takes the open unit ball of B/S into the closed unit ball of $E_1/j(E)$ and so $\|\tilde{U}\| \leq 1$.

More is so. If we take a typical member x of $E_1/j(E)$, then x is of the form

$$\begin{aligned} x &= \pi(b,e)+j(E) \\ &= \pi((b,0)+(0,e))+j(E) \\ &= \pi(b,0)+\pi(0,e)+j(E) \\ &= \pi(b,0)+j(e)+j(E) \\ &= \pi(b,0)+j(E) \\ &= \tilde{U}(b+S), \end{aligned}$$

so $\tilde{U}$ is surjective. Finally,

$$\begin{aligned} \|x\|_{E_1/j(E)} &= \inf_{e\in E}\left\{\|\pi(b,0)+j(e)\|_{E_1}\right\} \\ &= \inf_{e\in E}\left\{\|\pi(b,0)+\pi(0,e)\|_{E_1}\right\} \\ &= \inf_{e\in E}\{\left\|\pi(b,e)\right\|_{E_1}\} \\ &= \inf_{e\in E, s\in S}\left\{\|(b,e)+(s,-u(s)\|_{B\oplus_1 E}\right\} \\ &= \inf_{e\in E, s\in S}\left\{\|b+s\|+\|e-u(s)\|\right\} \\ &= \inf_{s\in S}\left\{\|b+s\|\right\} = \|b+S\|_{B/S}. \end{aligned}$$

In other words $\tilde{U}$, is an isometric isomorphism of B/S onto $E_1/j(E)$.

There's much that's magic in the above theorem of Kislyakov. Some is easily detected. For instance, suppose S, B, E and u are as in the hypotheses of the theorem. Imagine that u is also supposed to be an isomorphism with, say, $\|u(s)\| \geq \delta\|s\|$ for all $s \in S$, where $0 < \delta < 1$. Then for any $b \in B$,

$$\begin{aligned} \|\tilde{u}(b)\|_{E_1} &= \|\tilde{u}(b)\|_{(B\oplus_1 E)/N} \\ &= \|\pi(b,0)\|_{(B\oplus_1 E)/N} \\ &= \inf_{s\in S}\left\{\|(b,0)+(s,-u(s))\|\right\} \\ &= \inf_{s\in S}\left\{\|b+s\|+\|u(s)\|\right\} \\ &\geq \inf_{s\in S}\left\{\delta\|b+s\|+\delta\|(s)\|\right\} \\ &= \delta\|b\|. \end{aligned}$$

It's a stunning fact (that's also useful) that Kislyakov's construction of E_1, j and $\tilde{u}$ really has but one possible outcome. To see why this is so we take a momentary detour to establish an abstract property pertaining to the construction, namely, if F is a Banach space, $w : B \to F$ and $v : E \to F$ are bounded linear operators with $vu = w|_S$, then there is a unique linear map $\varphi : E_1 \to F$ such that $v = \varphi\tilde{u}$ and $v = \varphi j$. Moreover, $\|\varphi\| \leq \max\left\{\|v\|, \|w\|\right\}$.

Pictorially,

$$\begin{tikzcd}
B \arrow[r, "\tilde{u}"] \arrow[rr, bend left, "w"] & E_1 \arrow[r, "\varphi"] & F \\
S \arrow[u, hook] \arrow[r, "u"] & E \arrow[u, "j"] \arrow[ur, "v"] &
\end{tikzcd}$$

φ? Well, no matter how you cut it, at a typical $(b, e) \in B \oplus_1 E, \varphi(\pi(b, e)$ must be $w(b) + v(e)$. After all

$$\varphi\big(j(e)\big) = v(e) \quad \& \quad \varphi\big(\tilde{u}(b)\big) = w(b)$$

so

$$\begin{aligned}
\varphi\big(\pi(b,e)\big) &= \varphi\big(\pi(b,0) + \pi(0,e)\big) \\
&= \varphi\big(\tilde{u}(b) + j(e)\big) \\
&= \varphi\big(\tilde{u}(b)) + \varphi\big(j(e)\big) \\
&= w(b) + v(e).
\end{aligned}$$

Regarding φ's norm, if $\pi(b, e)$ is a typical member of E_1 with $\pi(b, e) < 1$, then there must be a $b_0 \in B$ and an $e_0 \in E$ with $\|b_0\| + \|e_0\| (= \|(b_0, e_0)\|_{B \oplus_1 E}) < 1$ so that $\pi(b, e) = \pi(b_0, e_0)$. It follows that

$$\begin{aligned}
\big\|\varphi(\pi(b,e))\big\| &= \big\|\varphi(\pi(b_0,e_0))\big\| \\
&= \|w(b_0) + v(e_0)\| \\
&\leq \|w(b_0)\| + \|v(e_0)\| \\
&\leq \|w\| \, \|(b_0)\| + \|v\| \, \|e_0\| \\
&\leq \max\{\|w\|, \|v\|\} \, (\|b_0\| + \|e_0\|) \\
&< \max\{\|v\|, \|w\|\}.
\end{aligned}$$

This fact allows us to establish the uniqueness of the triple $(E_1, j, B \xrightarrow{\tilde{u}} E_1)$. Stated formally this goes as follows:

Uniqueness of Kislyakov's Construct: The triplet $(E_1, j, \tilde{u})$ is unique in the following sense: if $(E_1', j', \tilde{u'})$ is another triplet such that the following diagram commutes

$$\begin{tikzcd}
B \arrow[r, "\tilde{u}'"] & E_1' \\
S \arrow[u, hook] \arrow[r, "u"] & E \arrow[u, "j'"]
\end{tikzcd}$$

where E_1' is a Banach space, $j' : E \to E_1'$ is an isometric embedding, $\tilde{u}' : B \to E_1'$ is a bounded linear operator with $\tilde{u}'|_S = j'u$ and $(E_1', j', \tilde{u'})$ satisfy the characteristic property established above [that is, given a Banach space F and bounded linear operator $w : B \to F, v : E \to F$ such that $vu = w|_S$, there is a unique linear map $\varphi' : E_1' \to F$ such that $w = \varphi' \tilde{u}'$ and $v = \varphi' j'$], then there is an isometric isomorphism $T : E_1 \to E_1'$ of E_1 onto E_1' such that $Tj = j'$.

Phew!

The proper formulation of uniqueness is almost longer that its proof.

As a matter of fact, if we start with $(E_1, j, \tilde{u})$, let $F = E_1'$, $w = \tilde{u}'$ and $v = j'$, then we get a unique linear operator J from E_1 to E_1' such that commutes. On the other hand, if we take $F = E_1$, $w = \tilde{u}$

$$\begin{tikzcd}
B \arrow[r, "\tilde{u}"] \arrow[rr, bend left, "\tilde{u}'"] & E_1 \arrow[r, "T"] & E_1' \\
S \arrow[u, hook] \arrow[r, "u"] & E \arrow[u, "j"] \arrow[ur, "j'"'] &
\end{tikzcd}$$

and $v = j$, then applying the characteristic mumbo-jumbo to the triple $(E_1', j', \tilde{u}')$ we get a unique linear operator $T' : E_1' \to E$ such that commutes.

$$\begin{tikzcd}
B \arrow[r, "\tilde{u}'"] \arrow[rr, bend left, "\tilde{u}"] & E_1' \arrow[r, "T'"] & E_1 \\
S \arrow[u, hook] \arrow[r, "u"] & E \arrow[u, "j'"] \arrow[ur, "j"'] &
\end{tikzcd}$$

Take a deep breath and realize that if we finally take the triple $(E_1, j, \tilde{u})$ and for F we take E, for w we take $\tilde{u}$ and for ν we take j, then id_E and $T'T$ both work as φ; the uniqueness of φ says $T'T = id_{E_1}$. Turning things around, we see that $T'T = id_{E_1}$, with each of T, T' having norm ≤ 1. This is tantamount to establishing our claims.

In keeping with the Bourgain-Pisier game plan, we say that the embedding $j : E \to E_1$ $\big[$for which there is an operator $\tilde{u} : B \to E_1$ such that $\|\tilde{u}\| \leq 1$ and $\tilde{u}|_S = ju\big]$ is associated with (E, u, S, B); sometimes we'll say that $(E_1, j, \tilde{u})$ is associated with (E, u, S, B).

Another simple observation.

Proposition 1 *Suppose E, u, S and B are as in Kislyakov's theorem and the isometric embedding j : $E \to E_1$ is associated with (E, u, S, B). Let N be a closed linear subspace of E; suppose $g : E \twoheadrightarrow E/N$ and $g_1 : E_1 \twoheadrightarrow E_1/j(N)$ are the natural quotient maps.*

Then the induced isometric embedding

$$\tilde{j} : E/N \to E_1/j(N)$$

is associated with $(E/N, gu, S, B)$.

In fact, take a look at the picture:

$$\begin{tikzcd} B \arrow[r, "\varpi"] & F \\ S \arrow[u, hook] \arrow[r, "gu"] & E/N \arrow[u, "v"'] \end{tikzcd}$$

where F is some Banach space, $w : B \to E$ and $v : E/N \to F$ are bounded linear operator and $w|_S = vgu$. By the characteristic property of $(E_1, j, \tilde{u})$ we know there is a unique linear operator $\varphi : E_1 \to F$ such that

$$\varphi j = vg \quad , \quad \varphi \tilde{u} = w$$

and $\|\varphi\| \le \max\{\|w\|, \|vg\|\}$. It's plain that φ vanishes on $j(N)$ — if $n \in N$, then $g(n) = 0$ in E/N, so $v(g(n)) = 0$; hence, there is an operator $\tilde{\varphi} : E_1/j(N) \to F$ such that $\varphi = \tilde{\varphi} g_1$,

$$\tilde{\varphi}\tilde{j} = v,$$

and

$$\|\tilde{\varphi}\| = \|\varphi\| \le \max\big\{\|w\|, \|v\|\big\}.$$

φ's uniqueness implies that of $\tilde{\varphi}$.

It follows now from the uniqueness of Kislyakov's construct that $\tilde{j}$ is associated with $(E/N, gu, S, B)$.

3. Bourgain and Pisier Get Serious

Let $0 < \eta \le 1$.

We say that an isometric embedding $j : E \to E_1$ is η-admissible if there exists a Banach space B and a bounded linear operator u from a closed linear subspace S of B to E with $\|u\| \le \eta$ and a bounded linear operator $\tilde{u}{:}B \to E_1$ so that $(E_1, j, \tilde{u})$ is associated with (E, u, S, B).

Fact An isometric embedding $j : E \to E_1$ is η-admissible if and only if there exists a Banach space B and a metric quotient $\pi : B \oplus_1 E \twoheadrightarrow E_1$ such that the following is so:

$$(*)\begin{cases} \text{For each } b \in B \text{ and } e \in E,\ \|\pi((b,e))\| \ge \|e\| - \eta\|b\| \\ \text{and } \pi((0,e)) = j(e). \end{cases}$$

After all, if j is η-admissible and $(E_1, j, \tilde{u})$ is associated with (E, u, S, B), where $\|u\| \le \eta$, then the metric quotient map $\pi : B \oplus_1 E \to E_1$ with kernel $N = \{(s, -us) : s \in S\}$ satisfies

$$\begin{aligned} \big\|\pi((b,e))\big\| &= \inf_{s \in S}\big\{\|b+s\| + \|e - us\|\big\} \\ &\ge \inf_{s \in S}\big\{\eta(\|s\| - \|b\|) + \|e\| - \eta\|s\|\big\} \\ &= \|e\| - \eta\|b\|. \end{aligned}$$

On the other hand, if $(*)$ is in effect, then whenever $(b, e) \in \ker(\pi)$,

$$0 = \|\pi(b,e)\| \ge \|e\| - \eta\|b\|,$$

so that

$$\eta\|b\| \le \|e\|.$$

If we let S be the image of the projection of $\ker\pi$ onto B, $S = \{b \in B : \pi((b,e)) = 0 \text{ for some } e \in E\}$, then whenever $s \in S$, there is an $e_s \in E$ so that $\pi((s,e_s)) = 0$. Be careful here! For each $s \in S$ there is an $e_s \in E$ so that $\pi((s,e_s)) = 0$ and this is a one-per-customer deal. If e_s and e_s' both satisfy

$$\pi((s,e_s)) = 0 = \pi((s,e_s')),$$

then

$$\pi((0,e_s - e_s')) = \pi((s,e_s)) - \pi(s,e_s')) = 0.$$

But then

$$0 = \pi((0,e_s - e_s')) = j(e_s - e_s')$$

which, by j's isometric character, forces $e_s = e_s'$. A natural map is borne: $s \to e_s$ from S to E, – call it "$-u$". It is plain and easy-to-see that $(E_1, j, \tilde{u})$, where $\tilde{u}(b) = \pi((b,0))$, is associated with (E,u,S,B).

It is noteworthy that if $j_0 : E_0 \to E_1, j_1 : E_1 \to E_2, \ldots, j_n : E_n \to E_{n+1}$ are each η-admissible embedding, then $j_n \circ j_{n-1} \circ \ldots \circ j_0 : E_0 \to E_{n+1}$ is an η-admissible embedding,too.

Indeed, if $j_0 : E_0 \to E_1$ is an η-admissible embedding, then it is because there's a Banach space B_0 and a metric quotient.

$$\pi_0 : B_0 \oplus_1 E_0 \twoheadrightarrow E_1$$

such that for each $b_0 \in B_0$ and $e_0 \in E_0$,

$$\big\|\pi_0((b_0,e_0))\big\| \ge \|e_0\| - \eta\|b_0\|$$

and

$$\pi_0((0,e_0)) = j_0(e_0).$$

Since $j_1 : E_1 \to E_2$ is also an η-admissible embedding, there must be a Banach space B_1 and a metric quotient

$$\pi_1 : B_1 \oplus_1 E_1 \twoheadrightarrow E_2$$

such that for each $b_1 \in b_1$ and $e_1 \in E_1$

$$\big\|\pi_1((b_1,e_1))\big\| \ge \|e_1\| - \eta\|b_1\|,$$

and

$$\pi_1((0,e_1)) = j_1(e_1).$$

Look at the metric quotient map

$$\pi : B_1 \oplus_1 B_0 \oplus_1 E_0 \twoheadrightarrow E_2$$

given by

$$\pi((b_1,b_0,e_0)) = \pi_1((b_1,\pi_0(b_0,e_0))).$$

Check it out:

$$\begin{aligned}
\big\|\pi((b_1,b_0,e_0))\big\| &= \Big\|\pi_1((b_1,\pi_0((b_0,e_0))))\Big\| \\
&\ge \|\pi_0((b_0,e_0))\| - \eta\|b_1\| \\
&\ge \|e_0\| - \eta\|b_0\| - \eta\|b_1\| \\
&= \|e_0\| - \eta(\|b_0\| + \|b_1\|) \\
&= \|e_0\| - \eta\|(b_1,b_0)\|_{B_1\oplus_1 B_0}
\end{aligned}$$

and

$$\begin{aligned}\pi((0,0,e_0)) &= \pi_1((0,\pi_0((0,e_0))) \\ &= j_2(\pi_0((0,e_0))) = j_1 j_0(e_0).\end{aligned}$$

A clear path is indicated.

Where's all this leading us?

Recall how the Banach space inductive limit of Banach paces is defined. Let $(E_n)_{n\geq 0}$ be a sequence of Banach spaces along with a sequence $j_n : E_n \to E_{n+1}$ of isometric embeddings. The inductive limit X of the system (E_n, j_n) is defined as follows: consider the linear subspace of ΠE_n formed by all sequences (x_n) such that $j_n x_n = x_{n+1}$ for all n sufficiently large; equip this space with the semi-norm

$$\|(x_n)\| = \lim \|x_n\|$$

and let $\mathfrak{X}$ be the normed linear space obtained after passing to the quotient by the kernel of this semi-norm.

The space $X = ind - lim(E_n, j_n)$ is the completion of the space $\mathfrak{X}$. It is easy-to-see that there is a system $J_n : E_n \to X$ of isometric embeddings such that if X_n is $J_n(E_n)$ then $X_n \subseteq X_{n+1}$ and $\cup_n X_n$ is dense in X.

Here's a remarkable result due to Bourgain and Pisier.

Theorem 2 *Let $0 < \eta \leq 1$. Suppose $(E_n)_{n\geq 0}$ is a sequence of finite dimensional Banach spaces and $j_k : E_k \to E_{k+1}$ ($k \geq 0$) is a sequence of η-admissible isometric embeddings.*

Then ind-lm (E_n, j_n) has the Radon-Nikodym property.

We delay the proof of Theorem 2 until section 4**.** We present instead a crucial (at least, for these deliberations) result that follows from it.

Theorem 3 *Let $\lambda > 1$ and E be any separable Banach space, then there is a separable $\mathscr{L}_{\infty,\lambda}$ space denoted by $\mathscr{L}_\lambda[E]$, which contains E isometrically, such that, $\mathscr{L}_\lambda[E]/E$ has the Radon-Nikodym property.*

PROOF. Let $(F_n)_{n\geq 0}$ be an increasing sequence of finite dimensional subspaces of E such that $\cup_n F_n$ is dense in E. Fix $\eta : \frac{1}{\lambda} < \eta < 1$. We will construct a sequence of η-admissible embeddings.

$$j_0 : E \to E_1, \ldots, j_n : E_n \to E_{n+1}, \ldots,$$

together with a sequence (G_n) of finite dimensional subspaces $G_n \subseteq E_n$ such that $G_0 = \{0\}$ and, for $n \geq 1$,

$$(j_{n-1} \circ \ldots \circ j_0)(F_{n-1}) \cup j_{n-1}(G_{n-1}) \subseteq G_n$$

and, for $n \geq 1$,

$$d(G_n, \ell^\infty_{dimG_n}) \leq \lambda$$

To start, fix $\varepsilon > 0$ such that $1 + \varepsilon = \lambda\eta > 1$.

Key to the construction is the fact that for any $\varepsilon > 0$ any finite dimensional space is $(1+\varepsilon)$-isomorphic to a subspace of ℓ^∞_m, m sufficiently large.

Start with F_0.

F_0 is $(1+\varepsilon)$-isomorphic to a subspace S of $\ell^\infty_{m_0}$, m_0 sufficiently large. So there is an isomorphism $u : S \to E$ of S into E so $\|u\| \leq \eta$ and $\|n|^{-1}_{F_0}\| \leq \lambda$. Apply Kislyakov's theorem to $(E, u, S, l^\infty_{m_0})$ to find a Banach space E_1, an isometric embedding $j_0 : E \to E_1$ and an operator $\tilde{u} : \ell^\infty_{m_0} \to E_1$ such that $\tilde{u}|_S = j_0 u$ and $\|\tilde{u}\| \leq 1$. As we noted immediately following Kislyakov's theorem, $\tilde{u}$ is, in fact, an isomorphism, too, with $\|\tilde{u}^{-1}|_{G_1 = \tilde{u}(\ell^\infty_{m_0})}\| \leq \lambda$.

So $d(G_1, \ell^\infty_{m_0}) \leq \lambda$.

Enlarge the scope for our construction.

Let $H = span\{j_o(F_1) \cup G_1\} \subseteq E_1$.

H is $(1+\varepsilon)$-isomorphic to a subspace S of $\ell^\infty_{m_1}, m_1$ sufficiently large. So there is an isomorphism $u : S \to E_1$ of S into E_1 so $\|u\| \le \eta$ and $\|u^{-1}|_H\| \le \lambda$. Apply Kislyakov's theorem to $(E_1, u, S, l^\infty_{m_1})$ to find a Banach space E_2, an isometric embedding $j_1 : E_1 \to E_2$ and an operator $\tilde{u} : \ell^\infty_{m_1} \to E_2$ such that $\tilde{u}|_S = j_1 u$ and $\|\tilde{u}\| \le 1$.

Again, $\tilde{u}$ is also an isomorphism and letting $G_2 = \tilde{u}(\ell^\infty_{m_1})$, $\|\tilde{u}^{-1}|_{G_2}\| \le \lambda$, ensuring $d(G_2, \ell^\infty_{m_1}) \le \lambda$.

Continue in this way to complete the construction of the E_n's, j_n's and G_n's.

Let X be the inductive limit of the system (E_n, j_n). We may as well consider (E_n) to be an increasing sequence of subspaces of X.

Let Y be the closure of $\cup_n G_n$ in X; Y is a $\mathscr{L}_{\infty,\lambda}$ -space and contains $\overline{\cup_n F_n} = E$.

What of Y/E?

Plainly, Y/E is naturally embedded in X/E, the inductive limit of the spaces E_n/E; moreover, by Proposition 1 and the opening discussion of this section the embedding of E_n/E into E_{n+1}/E is η-admissible for each $n \ge 1$. It follows that X/E has the Radon-Nikodym property and so, too, does Y/E. Let us now recall the elegant result of Gerry Edgar [[13], pp. 210-211] that if X is a Banach space and E is a closed linear subspace of X such that both E and X/E have the Radon-Nikodym property, then X itself has the Radon-Nikodym property. Starting with $E = \ell^2$ and applying the above theorem, we get the following:

Corollary 1 *There exist, for any $\lambda > 1$, a $\mathscr{L}_{\infty,\lambda}$ -space X containing ℓ^2 isometrically such that X has the Radon-Nikodym property.*

This X will occupy our attentions in the next sections. Before that, we need to provide a proof for Theorem 2.

4. A Detailed Proof of Theorem 2

Again,

Theorem 4 *Let $0 < \eta < 1$. Suppose that $E_0, E_1, \ldots$ are finite dimensional Banach spaces and let $j_0 : E_0 \to E_1, j_1 : E_1 \to E_2, \ldots$ be a sequence of η- admissible isometric embeddings.*

Then the inductive limit of the sequence (E_n, j_n) has the Radon-Nikodym property.

Fix $\delta > 0$ and let E be any Banach space. A subspace N of E is δ-well placed in E whenever

$$(*) \left\{ \begin{array}{l} \text{given a probability space } (\Omega, \Sigma, P) \text{ and a } Z \in L^1_E(P) \\ \text{such that } \int Z dP \in N \text{ we have} \\ \int \|Z\| dP \ge \|\int Z dP\| + \delta \int \|gZ\|_{E/N} dP, \\ \text{where } g : E \to E/N \text{ is the quotient map.} \end{array} \right.$$

The first thing we'll do is see what happens in the above set-up if $\int Z dP$ is not necessarily in N but near to N. Here's what's so:

$$(**) \int \|Z\| dP \ge \Big\| \int Z dP \Big\| + \delta \int \|gZ\| dP - (2+\delta) \Big\| g(\int Z dP) \Big\|,$$

the added 'fudge-factor' $(2+\delta)\big\|g(\int Z dP)\big\|$ effectively accounting for how far $\int Z dP$ is from N.

Let's see why $(**)$ is so. Regardless of how small $\varepsilon > 0$ is we can find $y \in N$ so that

$$\Big\| \int Z dP - y \Big\| \le \Big\| g(\int Z dP) \Big\| + \varepsilon$$

Now look at $\tilde{Z} = Z - \int ZdP + y$. Of course,

$$\int \tilde{Z}dP = y \in N$$

and so $(*)$ applies; the result

$$\begin{aligned}\int \|\tilde{Z}\|dP &\geq \Big\| \int \tilde{Z}dP Big\| + \delta \int \|g\tilde{Z}\|dP \\ &= \|y\| + \delta \int \|gZ - g(\int ZdP) + g(y)\|dP \\ &= \|y\| + \delta \int \|gZ - g(\int ZdP)\|dP.\end{aligned}$$

Turnabout is fair-play so $Z = \tilde{Z} + \int ZdP - y$ and

$$\int \|Z\|dP \geq \int \|\tilde{Z}\|dP - \Big\| \int ZdP - y\Big\|.$$

Since $\|y\| \geq \| \int ZdP\| - \| \int ZdP - y\|$ and $\int \|gZ - g(\int ZdP)\|dP \geq \int \|gZ\|dP - \|g(\int ZdP)\|$ we can list some features worthy of special mention

$$\int \|Z\|dP \geq \int \|\tilde{Z}\|dP - \Big\| \int ZdP - y\Big\|. \tag{1}$$

$$\int \|\tilde{Z}\|dP \geq \|y\| + \delta \int \Big\|gZ - g(\int ZdP)\Big\|dP. \tag{2}$$

$$\|y\| \geq \Big\| \int ZdP\Big\| - \Big\| \int ZdP - y\Big\| \tag{3}$$

$$\Big\| \int ZdP - y\Big\| \leq \Big\|g(\int ZdP)\Big\| + \epsilon \tag{4}$$

and

$$\int \Big\|gZ - g\Big(\int ZdP\Big)\Big\| \geq \int \|gZ\|dP - \Big\|g(\int ZdP)\Big\|. \tag{5}$$

Ready?

$$
\begin{aligned}
\int \|Z\| dP &\geq \int \|\tilde{Z}\| dP - \Big\| \int Z dP - y \Big\| \quad \text{by (1)} \\
&\geq \|y\| + \delta \int \Big\| gZ - g\Big(\int Z dP\Big)\Big\| dP - \Big\| \int Z dP - y \Big\| \quad \text{by (2)} \\
&\geq \|y\| + \delta \int \Big\| gZ - g\Big(\int Z dP\Big)\Big\| dP - \Big\| g\Big(\int Z dP\Big)\Big\| - \varepsilon \quad \text{by (4)} \\
&\geq \Big\| \int Z dP \Big\| - \Big\| \int Z dP - y \Big\| + \delta \int \Big\| gZ - g\Big(\int Z dP\Big)\Big\| dP - \\
&\quad \Big\| g(\int Z dP) \Big\| - \varepsilon \quad \text{by (3)} \\
&\geq \Big\| \int Z dP \Big\| - \Big\| \int Z dP - y \Big\| + \delta\Big(\int \|gZ\| dP - \Big\| g\Big(\int Z dP)\Big\|\Big) - \\
&\quad \Big\| g(\int Z dP)\Big\| - \varepsilon \quad \text{by (5)} \\
&= \Big\| \int Z dP \Big\| + \delta \int \|gZ\| dP - \Big\| \int Z dP - y \Big\| - \\
&\quad (1+\delta)\Big\| g\Big(\int Z dP\Big)\Big\| - \varepsilon \quad \text{by golly} \\
&\geq \Big\| \int Z dP \Big\| + \delta \int \|gZ\| dP - \Big(\Big\| g\Big(\int Z dP\Big)\Big\| + \varepsilon\Big) - \\
&\quad (1+\delta)\Big\| g\Big(\int Z dP\Big)\Big\| - \varepsilon \quad \text{by (3)} \\
&= \Big\| \int Z dP \Big\| + \delta \int \|gZ\| dP - (2+\delta)\Big\| g\Big(\int Z dP\Big)\Big\| - 2\varepsilon.
\end{aligned}
$$

Letting $\varepsilon \searrow 0$ give us $(**)$. A variation on $(**)$ is also of use. It involves conditioning.

So suppose $\sum_0$ is a sub-σ-algebra of $\sum$. Let $Z \in L_E^1$. Then we have, by arguments totally analogous to those that produced $(**)$, for almost sure,

$$\mathbb{E}(\|Z\| \mid \sum_0) \geq \|\mathbb{E}(Z|\Sigma_0)\| + \delta\mathbb{E}(\|gZ\| \,|\Sigma_0) - (2+\delta)\|g(\mathbb{E}(Z|\Sigma_0))\|.$$

On integrating we get for any sub-σ-algebra $\sum_0$ of $\sum$ and any $Z \in L_E^1$ the following

$$(***) \int \|Z\| dP \geq \int \|\mathbb{E}(Z|\Sigma_0)\| dP + \delta \int \|gZ\| dP - (2+\delta) \int \|g(\mathbb{E}(Z|\Sigma_0))\| dP.$$

Now we're set for the main technical lemma the Bourgain-Pisier presentation.

Lemma 1 *Let $0 < \eta < 1$ and $\delta = \frac{1-\eta}{1+\eta}$. If N is δ-well placed in E and $j : E \to E_1$ is η-admissible, then $j(N)$ is δ-well placed in E_1*

η-admissibility of $j : E \to E_1$ hints that there is a Banach space B, a subspace S of B, a bounded linear operator $u : B \to E$ with $\|u\| \leq \eta$ and a $\tilde{u} : B \to E_1$ so that $(E_1, j, \tilde{u})$ is associated with (E, u, S, B).

Recall that Kislyakov's construct led us to

$$E_1 = (B \oplus_1 E)/\{(s, -u(s)) : s \in S\},$$

$\pi : B \oplus_1 E \twoheadrightarrow E_1$ and $j(e) = \pi(0, e)$ for $e \in E$. To test $j(N)$ we let $Z_1 \in L_{E_1}^1$ be such that $\int Z_1 dP \in j(N)$.

Let $\varepsilon > 0$

Find $Z' \in L^1_B$ and $Z'' \in L^1_E$ so that for any $w \in \Omega$

$$Z_1(w) = \pi(Z'(w), Z''(w))$$

and

$$\|Z'(w)\| + \|Z''(w)\| \le (1+\varepsilon)\|Z_1(w)\|.$$

How to locate Z', Z''? Well, keep in mind that we're dealing with the projective tensor norm so

$$L^1(P)\hat{\otimes}(B \oplus_1 E) = L^1_{B\oplus_1 E}(P)$$

and

$$L^1(P)\hat{\otimes}E_1 = L^1_{E_1}(P).$$

Also, $\pi : B \oplus_1 E \twoheadrightarrow E_1$ is a quotient operator, an isometric quotient operator; it follows that π induces such an operator from $L^1_{B\oplus_1 E}(P)$ onto $L^1_{E_1}(P)$.

If Z were simple, then the nature of this induced quotient map makes it simple to see how $Z' \in L^1_B(P)$ and $Z'' \in L^1_E(P)$ are chosen. Indeed, if $Z(w)$ is identically z for $w \in A \in \Sigma$, then z must be $\pi(b,e)$ for some $b \in B$ and $e \in E$ with $\|b\| + \|e\| < (1+\epsilon)\|z\|$; let Z' be constantly b on A and Z'' be constantly e on A. For general $Z \in L^1_{E_1}(P)$ we appeal to Pettis's Measurability Theorem, a friend indeed, when there's a need.

Pettis's Measurability Theorem tell us that Z can be represented in the form $Z = \sum_n z_n \chi_{An}$ when the series converges absolutely in E_1, P-almost surely and $\sum \|z_n\|P(A_n)$ is as near to $\int \|Z\|dP$ as one would like. Now backtracking through π to $b_n \in B, e_n \in E$ with $\pi(b_n, e_n) = z_n$ and $\|b_n\| + \|e_n\| < (1+\epsilon)\|z_n\|$ is easy business indeed. The absolute convergence of $\sum_n z_n \chi_{An}(w)$ for $w \in \Omega$ soon leads to that of both $\sum_n b_n \chi_{An}(w)$ and $\sum_n e_n \chi_{An}(w)$ and with it to the definitions of $Z' \in L^1_B(P)$ and $Z'' \in L^1_E(P)$ such that $Z = \pi(Z', Z'')$.

Okay, with Z_1, Z', Z'' in hand, knowing that $\int Z_1 dP \in j(N)$ there must be an $n \in N$ so that $j(n) = \int Z_1 dP \in j(N)$. N is well placed in E so $\pi(0,n) = j(n)$. But $\int Z_1 dP = \pi(\int Z' dP, \int Z'' dP)$ so $\pi(\int Z' dP, \int Z'' dP) = \pi(0,n) = 0$ in E_1.

So there must be an $s \in S$ so that

$$\int Z' dP = s, \int Z'' dP - n = -u(s);$$

ah ha: $Z'' + us \in E$ satisfies

$$\int (Z'' + u(s))dP = \int Z'' dP + u(s) = n \in N.$$

But N is δ -well placed in E so

$$\int \|Z'' + us\|dP \ge \left\| \int (Z'' + us)dP \right\| + \delta \left\| \int g(Z'' + us) \right\| dP = \|n\| + \delta \int \|g(Z'' + us)\|dP.$$

Since

$$\|s\| = \left\| \int Z' dP \right\| \le \int \|Z'\|dP$$

we must have

$$\begin{aligned}
\int \|Z''\|dP &\ge \int \|Z'' + us\|dP - \|us\| \\
&\ge \int \|Z'' + us\|dP - \eta\|s\| \\
&\ge \int \|gZ''\|dP - \eta \int \|Z'\|dP.
\end{aligned}$$

To summarize,

$$\begin{aligned}\int \|Z''\| dP &\geq \int \|Z'' + us\| dP - \eta \int \|Z'\| dP \\ &\geq \|n\| + \delta \int \|g(Z' + us)\| dP - \eta \int \|Z'\| dP \\ &\geq \|n\| + \delta\Big(\int \|gZ''\| dP - \eta \int \|Z'\| dP\Big) - \eta \int \|Z'\| dP \\ &\geq \|n\| + \delta \int \|g(Z'')\| dP - (\eta + \delta\eta) \int \|Z'\| dP.\end{aligned}$$

Our choices of Z' and Z'' leave us with

$$\begin{aligned}(1+\epsilon)^{-1} \int \|Z_1\| dP &\geq \int \|Z''\| dP + \int \|Z'\| dP \\ &\geq \|n\| + \delta \int \|gZ''\| dP - (\eta + \delta\eta) \int \|Z'\| dP + \int \|Z'\| dP \\ &= \|n\| + \delta \int \|gZ''\| dP + (1 - \eta - \delta\eta) \int \|Z'\| dP.\end{aligned}$$

But $\delta = \frac{1-\eta}{1+\eta}$ and $\|n\| = \|jn\| = \|\int Z_1 dP\|$ so

$$\frac{1}{1+\epsilon} \int \|Z_1\| dP \geq \Big\| \int Z_1 dP \Big\| + \delta \Big(\int \|Z'\| dP + \int \|gZ''\| dP \Big).$$

If we notice though that $g_1 : E_1 \twoheadrightarrow E_1/j(N)$ is the natural quotient, then

$$\|g_1 Z_1\| \leq \|Z'\| + \|gZ''\|$$

So, in fact,

$$\frac{1}{1+\epsilon} \int \|Z_1\| dP \geq \Big\| \int Z_1 dP \Big\| + \delta \int \|g_1 Z_1\| dP.$$

If we now let $\epsilon \searrow 0$, then

$$\int \|Z_1\| dP \geq \Big\| \int Z_1 dP \Big\| + \delta \int \|g_1 Z_1\| dP$$

results and with it we have $j(N)$ is δ-well placed in E_1.

NOW we're ready to prove Theorem 2.

We may as well assume the E_m's are ascending with $\cup_m E_m$ dense in X. We'll let $(M_n)_{n\geq 0}$ be an X-valued $L^1(P)$-bounded martingale adapted to the ascending sequence $(\sum_n)_{n\geq 0}$ of sub-σ-algebras of $\sum$ and we'll show that (M_n) is almost surely convergent. Each E_m is δ-well placed in X.

$$Let \quad g_m : X \twoheadrightarrow X/E_m$$

be the natural quotient map. The key to this proof is to show that

$$\lim_{m\to\infty} \quad \lim_{n\to\infty} \quad \int \|g_m(M_n)\|_{X/E_m} dP = 0.$$

Indeed, Doob's Maximal Inequality tells us that for any $\varepsilon > 0$,

$$\overline{\lim} \int_{[\sup_n \|M_n\| > \varepsilon]} (\|M_n\| - \varepsilon) dP \geq 0.$$

Hence,

$$\varepsilon P\Big[\sup_n \|M_n\| > \varepsilon\Big] \leq \sup_n \int \|M_n\| dP$$

ensuring that $(M_n)_{n\geq 0}$ is almost surely bounded and

$$\sup_n \|g_m(M_n)\| \searrow 0$$

almost surely. It follows that for almost all $w \in \Omega$ the set $\{M_n(w) : n \geq 0\}$ is relatively norm compact subject of $X --$ keep in mind, the E_n's are finite dimensional. X is separable and so there is a countable weak*-dense subset $D \subseteq X^*$. Because $(x^* M_n)_{n\geq 0}$ is a martingale for each $x^* \in D$, a simple comparison of topologies soon reveals that $(M_n)_{n\geq 0}$ is almost surely pointwise convergent in the norm topology of X. So the issue of establishing that

$$\lim_{m\to\infty} \quad \lim_{n\to\infty} \quad \int \|g_m(M_n)\|_{X/E_m} dP = 0$$

is paramount to our cause.

Now, the E_m's are getting bigger so the sequence $\big(\|g_m(M_n)\|\big)$ is descending in m. Also, for each m, $(\|g_m(M_n)\|)_{n\geq 0}$ is a submartingale and so $\big(\int \|g_m(M_n)\| dP\big)_n$ is ascending. It follows that each of the hoped-for limits involved in

$$\lim_{m\to\infty} \quad \lim_{n\to\infty} \quad \int \|g_m(M_n)\|_{X/E_m} dP$$

exists and is a monotone limit.

Notice that for any $Z \in L^1_X(P)$ that

$$\lim_{m\to\infty} \quad \int \|g_m(Z)\|_{X/E_m} dP = 0.$$

Why is this so? Well, a moment's reflection reveals that if $x \in \cup_m E_m$ then $x \in E_{m_0}$ for some m_0, and so for any $m \geq m_0$, g_m annihilates x. It follows that for any $\cup_m E_m$ -valued simple random variable Z, eventually $\|g_m(Z(w))\| = 0$ for all $w \in \Omega$. Since this is so,

$$\lim_m \int \|g_m(Z)\|_{X/E_m} dP = 0.$$

Bootstrapping to X-valued, Σ-simple random variables Z is easy and, from there, to general $Z \in L^1_X(P)$ is simple.

Okay, for $k \leq n$, $\mathbb{E}\big(M_n | \sum_k\big) = M_k$ almost all the time; so using the observation $(***)$, regarding conditioning, that follows our Main Lemma, we see that if $k \leq n$, then for any m

$$\int \|M_n\| dP \geq \int \|M_k\| dP + \delta \int \|g_m(M_n)\| dP - (2+\delta) \int \|g_m(M_k)\| dP.$$

Let $n \to \infty$ and get

$$\lim_n \int \|M_n\| dP \geq \int \|(M_k)\| dP + \delta \lim_m \|g_m(M_n)\| dP - (2+\delta) \int \|g_m(M_k)\| dP.$$

Now let $m \to \infty$ and get

$$\begin{aligned}
\lim_n \int \|M_n\| dP \quad &\geq \quad \int \|(M_k)\| dP + \delta \lim_m \lim_n \int \|g_m(M_n)\| dP \\
&\quad -(2+\delta) \lim_m \int \|g_m(M_k)\| dP \\
&= \quad \int \|(M_k)\| dP + \delta \lim_m \lim_n \int \|g_m(M_n)\| dP.
\end{aligned}$$

Now let $k \to \infty$; the result is

$$\lim_n \int \|M_n\| dP \geq \lim_k \int \|(M_k)\| dP + \delta \lim_m \lim_n \int \|g_m(M_n)\| dP.$$

It follows that

$$0 \geq \delta \lim_m \lim_n \int \|g_m(M_n)\| dP,$$

and with this, Theorem 2 is proved.

5. The Bourgain-Pisier $\mathscr{L}_\infty$-spaces

Theorem 5 *For each $\lambda > 1$ there is a $\mathscr{L}_{\infty,\lambda}$ -space X with the Radon-Nikodym property such that $X \hat{\otimes} X$ contains an isomorphic copy of c_0.*

PROOF. Start with $E = \ell^2$ and let $X = \mathscr{L}_\lambda[E]$. Corollary 1 tells us that X has the Radon-Nikodym property.

Of course, X also contains a copy of ℓ^2 and, this in mind, we let (e_n) be the unit coordinate vector basis of ℓ^2, sitting, as it does, inside of X.

Since X is separable, there is a isometric embedding J of X into $C[0,1]$; J carries ℓ^2 into $C[0,1]$, isometrically, as well. Take $u \in \ell^2 \otimes \ell^2$ and view u as a finite rank bounded linear operator from ℓ^2 to ℓ^2. $JuJ^* : C[0,1]^* \to C[0,1]$ corresponds to the member $(J \otimes J)(u) \in C[0,1] \otimes C[0,1]$.

$C[0,1]^*$ is an L^1-space and the weak*-weak continuous linear operator JuJ^* plainly factors though ℓ^2, so Grothendieck's inequality assures us that JuJ^* is absolutely summing with

$$\pi_1(JuJ^*) \leq K_G \|JuJ^*\| \leq K_G \|u\|.$$

As with any absolutely summing operator into $C[0,1]$, JuJ^* is integral with

$$i(JuJ^*) = \pi_1(JuJ^*).$$

A weak*-weak continuous finite rank operator like JuJ^* is, defines a member-in-good-standing of $C[0,1] \hat{\otimes} C[0,1]$, with the projective norm of said member the same as the nuclear norm of the associated operator JuJ^* which, by all that's approximable, is just the integral norm $i(JuJ^*)$. So, if $u = \sum_{i \leq n} a_i e_i \otimes e_i$, then

$$\begin{aligned} \|u\|_{C[0,1] \hat{\otimes} C[0,1]} &= i(JuJ^*) = \pi_1(JuJ^*) \\ &\leq K_G \|u\| = K_G sup_{i \leq n} |a_i|. \end{aligned}$$

By the same token

$$\begin{aligned} \|u\|_{C[0,1] \hat{\otimes} C[0,1]} &\geq \|u\|_{C[0,1] \check{\otimes} C[0,1]} \\ &= \|u\|_{\ell^2 \check{\otimes} \ell^2} \\ &= sup_{i \leq n} |a_i|. \end{aligned}$$

All's well and $(e_n \otimes e_n)$ spans an isomorphic copy of c_0 in $C[0,1] \hat{\otimes} C[0,1]$. But what of $X \hat{\otimes} X$? Well, here X's $\mathscr{L}_\infty$ -nature saves the bacon.

It is one of the most elegant characteristics of $\mathscr{L}_\infty$ -spaces (due to Lindenstrauss)that X is a $\mathscr{L}_\infty$ -space precisely when X^{**} is injective. It follows from this that if X is a $\mathscr{L}_\infty$ -space that's a subspace of Y, then $X \hat{\otimes} Z$ is (isomorphic to) a subspace of $Y \hat{\otimes} Z$.

Schematically, this goes as follows:

$X\hat{\otimes}Z$ is always a subspace of $X^{**}\hat{\otimes}Z^{**}$; if X^{**} is injective, then X^{**} is a complemented subspace of Y^{**} and so $X^{**}\hat{\otimes}Z^{**}$ is a complemented subspace of $Y^{**}\hat{\otimes}Z^{**}$. Checking carefully we see (supposing that X^{**} is Λ-injective) that if $u \in X \otimes Z$, then

$$\|u\|_{Y\hat{\otimes}Z} \leq \|u\|_{X\hat{\otimes}Z} = \|u\|_{X^{**}\hat{\otimes}Z^{**}} \leq \Lambda\|u\|_{Y^{**}\hat{\otimes}Z^{**}} = \Lambda\|u\|_{Y\hat{\otimes}Z},$$

where the "Λ" factor comes about because X^{**} is complemented in Y^{**} by a projection of norm $\leq \Lambda$ making $X^{**}\hat{\otimes}Z^{**}$ a complemented subspace of $Y^{**}\hat{\otimes}Z^{**}$ via a projection of norm no more than Λ.

This allows us to compute $\| \sum_{i\leq n} e_i \otimes e_i\|_{X\hat{\otimes}X}$: for any n,

$$\begin{aligned}
\sup_{i\leq n}|a_i| &\leq \Big\|\sum_{i\leq n} a_ie_i \otimes e_i\Big\|_{C[0,1]\hat{\otimes}C[0,1]} \\
&\leq \Big\|\sum_{i\leq n} a_ie_i \otimes e_i\Big\|_{X\hat{\otimes}X} \\
&\leq \Lambda^2\Big\|\sum_{i\leq n} a_ie_i \otimes e_i\Big\|_{C[0,1]\hat{\otimes}C[0,1]} \\
&\leq \Lambda^2 K_G sup_{i\leq n}|a_i|,
\end{aligned}$$

and $(e_n \otimes e_n)$ still spans a c_0. ■

6. Q. Bu (And Friends) Look On The Sunny Side

The examples of Bourgain and Pisier plainly set boundaries on the possible implication *"if X and Y are Banach spaces with the Radon-Nikodym property, then their projective tensor product $X\hat{\otimes}Y$ has the property, too"*. Naturally, before their examples saw the light of day many examples existed where the implication held.

The most general case seemed to be roughly that if X and Y were dual spaces with the Radon-Nikodym property and one had the approximation property, then their projective tensor product also enjoyed the Radon-Nikodym property.

In 2000, Qingying Bu [6] found a characterization of the sequences that lie in $\ell^p\hat{\otimes}X$(if $1 < p < \infty$) and from this it followed that if $1 \leq p < \infty$ and X has the Radon-Nikodym property, then $\ell^p\hat{\otimes}X$ has the property as well.

One advantage of knowing (quantitatively) which sequences were in $\ell^p\hat{\otimes}X$ was found in the fact that using this information, Bu was able to show that the natural inclusion

$$\ell^p\hat{\otimes}X \hookrightarrow \ell^p_X$$

is a **semi-embedding**, that, is an injective linear operator such that the image of the closed unit ball $B_{\ell^p\hat{\otimes}X}$ is closed in ℓ^p_X. Then a call to Bourgain and Rosenthal [2] was made and the stability in question easily established using the elegant feature of the Radon-Nikodym property uncovered by them that if X is a separable Banach space that admits of a semi-embedding into a Banach space with the Radon-Nikodym property, then X has the property, too.

Soon Bu extended this result to $L^p(\mu)\hat{\otimes}X$ and with Paddy Dowling expanded the applicability of his idea to $U\hat{\otimes}X$, where U has an unconditional basis; moreover, Bu and Dowling established a variety of other important isomorphic invariants (including the non-containment of a copy of c_0) that pass from U and X to $U\hat{\otimes}X --$ where U is supposed to have an unconditional basis. The results of Bu and Dowling were soon subsumed by using the notion of a Schauder decomposition.

Let X be a Banach space and $(X_n)_{n\geq 1}$ be a sequence of closed linear subspaces of X. We say $(X_n)_{n\geq 1}$ is a **Schauder decomposition** of X if for any $x \in X$ there is a unique sequence (x_n) such that $x_n \in X_n$ for each n and $x = \sum_n x_n = \lim_{n\to\infty} \sum_{k=1}^n x_k$.

Should $(X_n)_{n\geq 1}$ be a Schauder decomposition of X, then for each $m \geq 1$, the map $P_m : X \to X$ that takes $x = \sum x_n \quad (x_n \in X_n)$ to the unique $x_m \in X_m$ that is X_m's contribution to $\sum_n x_n = x$ is a bounded linear projection with range X_m. If $(X_n)_{n\geq 1}$ is a Schauder decomposition of X, then $R_m : X \to X$ is the operator $R_m(x) = x - \sum_{n=1}^m P_n x$.

If $(X_n)_{n\geq 1}$ is a Schauder decomposition of X, then we say that (X_n) is **boundedly complete** if whenever (x_n) is a sequence with $x_n \in X_n$ for each n and $\sup_n \left\| \sum_{n=1}^n x_n \right\| < \infty$, we have $\lim_{n\to\infty} \sum_{n=1}^n x_n$ exists; $(X_n)_{n\geq 1}$ is **shrinking** provided that given $x^* \in X^*$ we have

$$\lim_{n\to\infty} \sup\{|x^*(x)| : x = R_n x, \|x\| \leq 1\} = 0.$$

In a completely analogous manner to what happens with Schauder bases (1-dimensional Schauder decompositions, if you please) we have the following satisfying results.

Theorem 6 (B.L. Sanders) *Let $(X_n)_{n\geq 1}$ be a Schauder decomposition of X. $(X_n)_{n\geq 1}$ is shrinking if and only if $(P_n(X)^*)_{n\geq 1}$ is a Schauder decomposition of X^*.*

Working in considerably greater generality, N.J. Kalton [18] put the topping on Sanders' Theorem with the following.

Theorem 7 (Kalton) *The Schauder decomposition $(X_n)_{n\geq 1}$, of X is shrinking if and only if the decomposition $(P_n(X)^*)_{n\geq 1} = (X^*)_{n\geq 1}$ is a boundedly complete decomposition of X^*.*

Naturally we will often ask more of the components X_n of a Schauder decomposition; so if each X_n is finite dimensional then we call the decomposition a finite dimensional decomposition (or FDD, for short). Here we see clear and present evidence of added hypotheses giving more information, structural information, about the spaces under view. Suppose $(X_n)_{n\geq 1}$ is a boundedly complete FDD for X. If $H = \{x^* \in X^* : \lim_{n\to\infty} \|x^* - \sum_{k=1}^n P_k^* \lambda_k^*\| = 0\}$, then X is isomorphic to H^*; what's more $(P_n^*(H))_{n\geq 1}$ is shrinking FDD for H.

Here's a reworking of an old favorite (of N. Dunford and A.P. Morse [14]) that bears repeating.

Theorem 8 *Let X be a Banach space having a boundedly complete Schauder decomposition $(X_n)_{n\geq 1}$. Suppose each X_n has the Radon-Nikodym property. Then X has the Radon-Nikodym property, too.*

PROOF. We follows the excellent lead of Dunford and Morse by renorming X, if necessary, to make sure our ducks are lined up; we want to make sure that our decomposition is 'monotone', that is, that

$$\left\| \sum_{i=1}^n x_i \right\| \leq \left\| \sum_{i=1}^{n+1} x_i \right\|$$

whenever $x_i \in X_i$, $i \in \mathbb{N}$. Of course, this can be done by renorming X, if need be, replacing the original norm by

$$|||\sum_n x_n||| = \sup_k \|\sum_{i=1}^k x_i\|$$

for $x_i \in X_i$,$\sum_n x_n \in X$. $|||\cdot|||$ is equivalent to $\|\cdot\|$ and has the desired monotonicity.

So we can, and do, assume our Schauder decomposition of X is boundedly complete and monotone. Now the proof follows a natural course. Let (Ω, Σ, P) be a probability space and $F : \Sigma \to X$ be a P -continuous vector measure having finite variation $|F|$. For each $n \in \mathbb{N}$, let $F_n : \Sigma \to X_n$ be $P_n F$; it's plain that each F_n is a P-continuous X_n -valued vector measure of finite variation and so for each n we can find an $f_n \in L^1_{X_n}(P)$ such that for any $E \in \Sigma$

$$F_n(E) = \int_E f_n dP.$$

For each $n \in \mathbb{N}$ we can define $\tilde{f}_n \in L^1_{X_n}(P)$ by

$$\tilde{f}_n = \sum_{m=1}^{n} f_m$$

Letting $\tilde{F}_n : \Sigma \to X$ be defined by

$$\tilde{F}_n(E) = \sum_{m=1}^{n} F_m(E)$$

we soon see that for any $E \in \Sigma$

$$\|\tilde{F}_n(E)\| = \Big\| \sum_{m=1}^{n} F_m(E) \Big\| \leq \Big\| \sum_{n} F_n(E) \Big\| = \|F(E)\|;$$

from this it follows that the variation $|\tilde{F}_n|$ of $\tilde{F}_n$ satisfies

$$|\tilde{F}_n|(E) \leq |F|(E)$$

regardless of $E \in \Sigma$. Naturally,

$$\tilde{F}_n(E) = \int_E \tilde{f}_n dP$$

and so

$$\int_E \Big\| \sum_{m=1}^{n} f_m \Big\| dP = \int_E \|\tilde{f}_n\| dP = |\tilde{F}_n|(E) \leq |F|(E) \leq |F|(\Omega) < \infty.$$

But regardless of $w \in \Omega$ and $n \in \mathbb{N}$, we have

$$\Big\| \sum_{m=1}^{n} f_m(w) \Big\| \leq \Big\| \sum_{m=1}^{n+1} f_m(w) \Big\|$$

so the Monotone Convergence Theorem steps in to conclude that for each $E \in \Sigma$

$$\int_E \sup_n \Big\| \sum_{m=1}^{n} f_m \Big\| dP = \int_E \lim_n \Big\| \sum_{m=1}^{n} f_m \Big\| dP = \lim_n \int_E \Big\| \sum_{m=1}^{n} f_m \Big\| dP \leq |F|(\Omega) < \infty.$$

It follows that for P -almost all $w \in \Omega$, $\sup_n \big\| \sum_{m=1}^{n} f_m(w) \big\| < \infty$ so by the boundedly complete nature of the decomposition $(X_n)_{n \geq 1}$, the series $\Sigma_n f_n(w)$ converges in X (at least P-almost everywhere).

The function $\tilde{f} : \Omega \to X$ defined by

$$\tilde{f}(w) = \begin{cases} \sum_n f_n(w) & \text{, if } \sup_n \big\| \sum_{m=1}^{n} f_m(w) \big\| < \infty \\ 0 & \text{, otherwise} \end{cases}$$

is P-measurable and

$$\int \|\tilde{f}\| dP = \int \Big\| \sum_n f_n \Big\| dP \leq |F|(\Omega) < \infty,$$

and so $\tilde{f} \in L^1_X(P)$. Further, it's plain to see that

$$F(E) = \int_E \tilde{f} dP.$$

Now suppose X has a boundedly complete finite dimensional decomposition and let $P_n : X \to X$ be the bounded linear projection of X onto X_n. Then $P_n \otimes id_Y : X\hat{\otimes}Y \to X\hat{\otimes}Y$ is a bounded linear projection with $\|P_n \otimes id_Y\| = \|P_n\|$; it is an easy computation to deduce that $((P_n \otimes id_Y)(X\hat{\otimes}Y))_n$ forms a Schauder decomposition of $X\hat{\otimes}Y$ More is so and pertinent to this discussion. In fact, we have the following. ∎

Theorem 9 (Eve Oja) *$((P_n \otimes id_Y)(X\hat{\otimes}Y))_n$ is a boundedly complete Schauder decomposition of $X\hat{\otimes}Y$ whenever $(P_n(X))_n$ is a boundedly complete finite dimensional decomposition of X.*

If one will keep faith with the discussion earlier (about stability of the Radon-Nikodym property when one space has an unconditional basis), then the following is a consequence of that and Oja's Theorem.

Corollary 2 *If X is a Banach space with a boundedly complete finite dimensional decomposition and Y has the Radon-Nikodym property, then $X\hat{\otimes}Y$ has the Radon-Nikodym property, too.*

We rush to take note (as done in [9])that many spaces arising in non-commutative analysis have boundedly complete finite dimensional decompositions without even being subspaces of spaces with unconditional bases.

The last topic we discuss here differs from the earlier ones in that there is no approximation assumptions inherent to the subject matter. The objective is to discuss what happens when X is a Banach lattice and Y is a Banach space, each enjoying the Radon-Nikodym property. The end product: $X\hat{\otimes}Y$ has the Radon-Nikodym property, too. The analysis is (for the most part) the work of Qingying Bu and Pei-Kee Lin[BL] and so we give a sketch of their main steps with added details when we vary the treatment.

To start we recall some basic features of the Banach space theory of Banach lattices.

Banach lattices enjoying the Radon-Nikodym property are very special animals indeed. Though we do not use the results we feel obligated to mention that Bourgain and Talagrand showed [3] that a Banach lattices with the Krein-Milman property have the Radon-Nikodym property and, in a truly amazing piece of mathematics, Talagrand [27] showed that separable Banach lattices with the Radon-Nikodym property are duals (of Banach lattices even)!

Generally, a Banach lattice with the Radon-Nikodym property contains no isomorph of c_0 and so, with due thanks to Meyer-Nieberg, must be Dedekind σ-complete. An appeal to another old gem (this due to Lozanovskii and Mekler) reveals that such Banach lattices have σ-order continuous norms. In sum, a Banach lattice with the Radon-Nikodym property is Dedekind complete and has an order continuous norm. Such lattices are weakly sequentially complete, can be decomposed into unconditional (direct) sums of closed 'bands' with weak order units (that're positive elements in the lattice) and, so, the analysis of these Banach lattice can often be reduced to the study of these 'bands' — themselves order continuous Banach lattices with weak order units that enjoy the fruits of the Monotone Convergence Theorem and are norm one complemented in their second dual. All this is given careful exposition in [24] and in [M-N BL], as is what we say next.

Once it's known that a Banach lattice has an order continuous norm and weak order unit, Kakutani's famous representation theory of Banach lattices is available. The result: there is a probability space(Ω, Σ, P) such that the given Banach lattice X can be viewed as a Banach function space (aka, a Köthe function space) of measurable real-valued functions defined on Ω with

$$L^\infty(\mu) \subseteq X \subseteq L^1(\mu);$$

moreover, each inclusion is continuous and the duality of X with its dual X^* is given by integration. To put things in lattice-theoretic context, we denote by X' the Köthe dual of X, that is,

$$X' = \Big\{g \in L^0(\mu) : \int |fg| d\mu < \infty \; for \; each \; f \in X\Big\},$$

where $L^0(\mu)$ denotes the linear space of measurable functions.

Under our working hypotheses (that X be a Banach lattice with the Radon-Nikodym property and with a weak order unit), it is well-know that $X' = X^*$. Keep in mind that X'' also makes sense but X'' need not be X^{**}!

For a given Banach space Y we denote by $X(Y)$ the linear space of all strongly μ -measurable Y-valued functions on Ω such that $\|f(\cdot)\|_Y \in X$; equip $X(Y)$ with the norm.

$$\|f\|_{X(Y)} = \left\|\|f(\cdot)\|_Y\right\|_X;$$

with the usual provisos and conventions in place, $X(Y)$ is a Banach space.

Also important to our cause is the space

$$X^*_{weak^*}(Y^*)$$

of all strongly μ -measurable $g : \Omega \to Y^*$ such that $g(\cdot)(y) \in X^*$ for each $y \in Y$; we norm $X^*_{weak^*}(Y^*)$ by

$$\|g\|_{X^*_{weak^*}(Y^*)} = \sup_{y \in B_Y} \|g(\cdot)(y)\|_{X^*}$$

One last definition. $X\langle Y\rangle$ a strongly μ-measurable function $f : \Omega \to Y$ belongs to $X\langle Y\rangle$ if for each $g \in X^*_{weak^*}(Y^*)$, $g(\cdot)(f(\cdot)) \in L^1(\mu)$ and equip $X\langle Y\rangle$ with the norm:

$$\|f\|_{X\langle Y\rangle} = \sup\left\{\left\|g(\cdot)(f())_{L^1(\mu)}\right\| : g \in B_{X^*_{weak^*}(Y^*)}\right\}$$

$X\langle Y\rangle$, with this norm, is a Banach space.

A few words about the work of Bu and Lin. Here's a fact of general interest.

Lemma 2 (Bu/Lin) *Let $f : \Omega \to Y$ be strongly μ-measurable and $\varepsilon > 0$*

Then there is a strongly μ-measurable $g_\varepsilon : \Omega \to Y^*$ such that $g_\varepsilon(w) \in B_{Y^*}$ for μ-almost all $w \in \Omega$ and satisfies

$$\|f(w)\| \leq |g_\varepsilon(w)(f(w))| + \varepsilon$$

for μ almost all $w \in \Omega$.

The proof is a nifty application of Pettis's Measurability Theorem.

Next, $X\langle Y\rangle$ and $X''(Y)$ are related.

Lemma 3 (Bu/Lin) *$X\langle Y\rangle \subseteq X''(Y)$ with $\|f\|_{X''(Y)} \leq \|f\|_{X\langle Y\rangle}$ whenever $f \in X\langle Y\rangle$. What's more, if $f_n \in B_{X\langle Y\rangle}$ and $f \in X''(Y)$ with $\lim_n \|f - f_n\|_{X''(Y)} = 0$, then $f \in B_{X\langle Y\rangle}$.*

Again, Pettis's Measurability Theorem plays a key role in the proof. One particularity relevant interpretation is worthy of mention: Lemma 3 says that the inclusion of $X\langle Y\rangle$ into $X(Y)(\subseteq X''(Y))$ is a semi-embedding, surely music to the ears of 'RNP fans'.

Key to the Bu/Lin paper is their representation of $X\hat{\otimes}Y$, when X is a Banach lattice having the Radon-Nikodym property and a weak order unit and Y is separable Banach space. Of course, we follow their lead and view X as a Köthe space with $X^* = X', X'' = X$ and X norm one complemented in X^{**}. All this in hand, Bu and Lin show $X\hat{\otimes}Y$ is isometrically isomorphic to $X\langle Y\rangle$.

This result is a *bit* more general than that contained in Bu and Lin [10] and we'll provide a proof that follows their lead with small detours taken to use Y's separability fully.

Define $\psi : X\hat{\otimes}Y \to X\langle Y\rangle$ by $\psi(z) = \sum_n x_n(\cdot)y_n$ whenever $z = \sum_n x_n \otimes y_n \in X\hat{\otimes}Y$; ψ is well-defined and $\|\psi(z)\|_{X\langle Y\rangle} \leq \|z\|_{X\hat{\otimes}Y}$.

Now let $f \in X\langle Y\rangle$.

Let $K = \beta((B_{X^{**}}, weak^*) \times B_Y)$, when βS denotes the Čech-Stone compactification of S.

Define

$$J : X^*_{weak*}(Y^*) \to C_b((B_{X^{**}}, weak*) \times B_Y)$$

$\big[$here the "b" denotes bounded$\big]$ by

$$Jg = x^{**}(g(\cdot)(y))$$

J is well-defined and $\|Jg\|_{C_b} = \|g\|_{X^*_{weak^*}(Y^*)}$. Keep in mind that $C_b((B_{X^{**}}, weak^*) \times B_Y) = C(K)$.

Now define F_f on J's range by

$$F_f(Jg) = \int g(t)(f(t))d\mu(t)$$

and realize that $F_f \in J(X^*_{weak^*}(Y^*))^*$ with $\|F_f\| = \|f\|_{X\langle Y\rangle}$.

Extend F_f using the Hahn-Banach theorem to an $\tilde{F}_f \in C(K)^*$; by the Riesz theorem, $\tilde{F}_f$ corresponds to a regular Borel measure v on K via

$$\tilde{F}_f(\varphi) = \int \varphi dv,\ \varphi \in C(K)$$

with $\|\tilde{F}_f\| = |v|(K)$.

Define $h_1 : K \to X^{**}, h_1(x^{**}, u) = x^{**}$ to any $(x^{**}, u) \in K$; h_1 is weak*-continuous and so is Gelfand integrable with respect to v.

Define $h_2 : (B_{X^{**}}, weak^*) \times B_Y \to B_Y$ by $h_2(x^{**}, y) = y$; h_2 is continuous. Let $j_Y : Y \to Y^{**}$ be the canonical inclusion. Then

$$j_Y h_2 : (B_{X^{**}}, weak^*) \times B_Y \to (B_{Y^{**}}, weak^*)$$

is also continuous and so extends uniquely to a continuous function $H_2 : \beta((B_{X^{**}}, weak^*) \times B_Y) \to (B_{Y^{**}}, weak^*)$. But it's easy to see that

$$\beta((B_{X^{**}}), weak^*) \times B_Y) = ((B_{X^{**}}), weak^*) \times \beta B_Y = K$$

and so H_2 takes K in a continuous fashion to $((B_{Y^{**}}), weak^*)$.

Now B_Y is Polish so $(B_{X^{**}}, weak^*) \times B_Y$ is v-measurable and

$$G_2 = H_2 \cdot \chi_{(B_{X^{**}}, weak^*) \times B_Y}$$

is scalarly measurable and has a separable range. Pettis's Measurability Theorem informs us that G_2 is strongly measurable and even Bochner integrable.

Write G_2 in the form

$$G_2 = \sum_n \chi_{B_n} y_n,$$

where (B_n) is a sequence of Borel's sets in K, $(y_n) \subseteq Y$ and (if $\varepsilon > 0$ is provided)

$$\sum \|y_n\|\ |v|(B_n) \leq \int \|G_2\| d|v| + \varepsilon \leq |v|(K) + \varepsilon.$$

Now for any $g \in X^*_{weak^*}(Y^*)$ we have

$$\begin{aligned} F_f(Jg) &= \int_K Jg(^{**}, u) dv(x^{**}, u) \\ &= \int_\Omega g(t)(f(t)) d\mu(t). \end{aligned}$$

Take $x^* \in X^*$ and $y^* \in Y^*$ and let $g = x^* y^*$; then

$$\begin{aligned}
\int x^*(y^* f(t) d\mu(t) &= \int_K h_1(x^{**}, u)(x^*) y^*(G_2(x^{**}, u)) dv(x^*, u) \\
&= \int_k h_1(x^{**}, u)(x^*) \sum_n y^*(y_n) \chi_{B_n}(x^{**}, u) dv(x^{**}, u) \\
&= \sum_n \int_{B_n} y^*(y_n) h_1(x^{**}, u)(x^*) dv(x^{**}, u) \\
&= \sum_n y^*(y_n) x_n^{**}(x^*),
\end{aligned}$$

where

$$x_n^{**} = Gelfand - \int_{B_n} h_1 dv.$$

Notice that for each $x^* \in X^*$ and $n \geq 1$

$$\begin{aligned}
|x_n^{**}(x^*)| &= \Big| \int_{B_n} h_1(x^{**}, u)(x^*) dv(x^{**}, u) \Big| \\
&\leq \int_{B_n} \|h_1(x^{**}, u)\| \quad \|x^*\| d|v|(x^{**}, u) \\
&\leq \|x^*\| \; |v|(B_n)
\end{aligned}$$

so $\|x_n^{**}\|_{X^{**}} \leq |v|(B_n)$. Now we have

$$\begin{aligned}
\sum_n \|y^*(y_n) x_n^{**}\|_{X^{**}} &\leq \sum_n \|y^*\| y_n\| \; \|x_n^{**}\| \\
&\leq \|y^*\| \sum_n \|g_n\| |v|(B_n) \\
&\leq \|y^*\| \big(|v|(K) + \varepsilon\big),
\end{aligned}$$

so $\sum_n y^*(y_n) x_n^{**}$ converges absolutely in X^{**}. Since the norm of X is order continuous, $X' = X^*$ and we know from Bu/Lin Lemma 2 that $f \in X\langle Y \rangle \subseteq X(Y)$ so for each $y^* \in Y^*$, $y^* f \in X$ and

$$y^*(x^* f) = x^*(y^* f) = \sum_n y^*(y_n) x_n^{**}(x^*);$$

it follows that

$$y^* f = \sum_n y^*(y_n) x_n^{**}.$$

If we denote by P the norm-one projection $P : X^{**} \to X$ and let $x_n = P x_n^{**}$, then $z = \sum_n x_n \otimes y_n \in X \hat{\otimes} Y$ with

$$\begin{aligned}
\|z\|_{X \hat{\otimes} Y} &\leq \sum \|x_n\| \quad \|y_n\| \\
&= \sum \|P x_n^{**}\| \quad \|y_n\| \\
&\leq \|P\| \sum_n \|y_n\| \quad \|x_n^{**}\| \\
&\leq \|P\| \sum_n \|y_n\| \quad |v|(B_n) \\
&\leq \|P\| \Big(|v|(K) + \varepsilon\Big) \\
&= \|P\| \Big(\|f\|_{X\langle Y \rangle} + \varepsilon\Big).
\end{aligned}$$

Let ε tend to zero and

$$\|z\|_{X\hat{\otimes}Y} \leq \|P\| \quad \|f\|_{X\langle Y\rangle} = \|f\|_{X\langle Y\rangle}$$

remains. Of course, $y^*f \in X$ and

$$y^*f = P(y^*f) = \sum_n y^*(y_n)P(x_n^{**}) = \sum_n y^*(y_n)x_n.$$

As before,

$$\begin{aligned} \Big\| \sum x_n(\cdot)y_n \Big\|_{X(Y)} &\leq \sum_n \|x_n(\cdot)\|_X \|y_n\| \\ &\leq \|P\|\big(\|f\|_{X\langle Y\rangle} + \varepsilon\big) \end{aligned}$$

and so $\sum_n x_n(\cdot)y_n \in X(Y)$ and, since $f \in X(Y)$, $f(\cdot) = \sum_n x_n(\cdot)y_n$, μ-almost everywhere, $f = \psi(z)$ and ψ is onto with

$$\|\psi(z)\|_{X\langle Y\rangle} \leq \|z\|_{X\hat{\otimes}Y} \leq \|P\| \; \|\psi(z)\|_{X\langle Y\rangle}.$$

All done.

What remains? Well, we need to call on a result of Bukhvalov [11] which says that if X is a Köthe function space with the Radon-Nikodym property and Y is a Banach space with the Radon-Nikodym property, then $X(Y)$ has the Radon-Nikodym property, as well.

Naturally a precursor to this is the classical result of Turett and Uhl [28] which assures us that L_X^P has the Radon-Nikodym property whenever X does and $1 < p < \infty$.

7. Concluding Remarks

We are dealing with the *projective* tensor product and so there is little access to subspace structure. This was the main point of several questions of **Bill Johnson**, asked of **Paddy Dowling** at the annual meeting of the AMS in Baltimore several years ago.

What can be said about the projective tensor product of a *subspace* X of $L^p(0,1)$, p bigger than 1, with a space Y having the Radon-Nikodym property? Does it have the property?

More generally, what can be said about the projective tensor product of a *subspace* of a Banach lattice with Radon-Nikodym property and a general Banach space with the property? Does it also have the Radon-Nikodym property?

Again, does the projective tensor product of a superreflexive Banach space with a space with the Radon-Nikodym property have the property?

Again, in much the same mode as the work of Bu and Dowling [8], many of the results that appear herein for spaces with the Radon-Nikodym have been generalized in the paper of Bu and Diestel [7].

One upshot of this progression of understanding of the stability of the Radon-Nikodym-like properties was the realization that it's entirely possible that for large classes of Banach spaces, having cotype is stable for the projective tensor product.

Acknowledgement. We'd like to expend special thanks to many who've talked to us about the varied topics that are involved in this survey. We extend special thanks to *Qingying Bu, Changsun Choi, Mienie Dekock, Paddy Dowling, Eve Oja, Daniele Puglisi, David Perez-Garcia* and the great cartographer, *Nacho Villanueva*.

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