RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **97** (1), 2003, pp. 1–12
Álgebra / Algebra

Isogroups and isosubgroups

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Abstract. The main goal of this paper is to give a mathematical foundation, serious and consistent, to some parts of *Santilli's isotheory*. We study the isotopic liftings of groups and subgroups and we also deal with the differences between an isosubgroup and a subgroup of an isogroup. Finally, some links between this isotheory and the standard groups theory, referred to representation and equivalence relations among groups are shown.

Isogrupos e isosubgrupos

Resumen. El principal objetivo de este artículo es proporcionar un fundamento matemático, consistente y riguroso, a determinadas partes de la *isoteoría de Santilli*. En él se realiza el levantamiento isotópico de los grupos y subgrupos, estudiándose asimismo la diferencia entre un isosubgrupo y un subgrupo de un isogrupo. Se muestran finalmente algunas relaciones entre esta isoteoría y la teoría standard de grupos, referentes a los temas de representación de grupos y de relaciones de equivalencia entre grupos.

1. Introduction

In 1978, the Italian-American theoretical physicist and mathematic Ruggero Maria Santilli proposes a generalization of conventional Lie's theory by using the concept of *isotopy* (in the Greek sense of being "axiom-preserving", also called *isotopic lifting*), which implies the origin of the actually known like *Santilli's isotheory* (see [2]). To do this, he extends the basic unit I=$(+1$, $diag(+1,...,+1)$, $...)$ of the initial structure to a generalized unit $\widehat{I}=\widehat{I}(x,\dot{x},\ddot{x},...,\mu,\tau,...)$, called *isounit*, which depends on the coordinate x and their derivatives, on the density μ, on the temperature τ and, in general, on any magnitude of the physic environment of the system in which we are. By using it, Santilli does a step-by-step generalization of the more important mathematical structures, obtaining other new ones, characterized by the fact of having the same properties as the initial ones, while the new units satisfy more general conditions than the verified by the initial ones. Santilli gives the name of *mathematical isostructures* to these new structures. In this way, he studies *isogroups*, *isorings*, *isofields*, *isovectorspaces* and *isoalgebras* (see [3], [4], [5] and [11], for instance).

It allowed him to get in a fast way some development of physical applications, principally in Quantum Mechanics and Dynamical Problems of particles and antiparticles. Santilli's isotopies allow to map any given and fixed linear, local and canonical structure into its most general possible non-linear, non-local

Presentado por Jesús Ildefonso Díaz.
Recibido: 6 de Febrero de 2002. *Aceptado*: 30 de Octubre de 2002.
Palabras clave / Keywords: Isotheory, Lie-Santilli, isogroup, isosubgroup
Mathematics Subject Classifications: 17B99.
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and non-canonical forms which are capable of reconstructing linearity, locality and canonicity in certain generalized isospaces and isofields within the fixed inertial coordinates of the observer.

However, in the last years, Santilli has found some mathematical inconsistencies in his early formulation of the isotheory. Due to it, Santilli and other mathematicians have studied the isotopic liftings of Functional Analysis and Differential Calculus (see [1] and [6]). It has allowed to get some important applications in Physics (see [7], [8], [9] and [10], for instance).

So, as Santilli's isotheory needs even a consistent mathematical foundation, Santilli himself has proposed several subjects of research to the international mathematical-scientific community. One of them consists on proving the existence of isostructures corresponding to the lifting of structures already known, although they do not have a practical application in Physic. Santilli thinks that it would be good to convince the scientific about the relevance of his research, which would give bigger consistence and reliance to his isotheory.

In this paper, we try to partially response to Santilli's petition, by studying a possible lifting of the simplest algebraic structure: the group structure. However, as there is already some studies about it (see [11]), we complete them, give also some examples and show how the subgroups can be isotopically lifted by using the Santilli's model to construct isoproducts. Getting this last question is the main goal of this paper.

To do this, we previously give in Section 2 some basic definitions related to isotopic liftings. Section 3 is devoted to the study of isogroups. Next, in Section 4, we obtain Theorem 2, which assures the construction of an isosubgroup, giving some examples, too. We also distinguish between subgroups and isosubgroups of an isogroup. Last two sections are devoted to study some applications of Santilli's isotheory: representation and equivalence relations among isogroups, respectively, and some links among these concepts and the standard groups theory.

2. Preliminaries

Remember that for a given and fixed mathematical structure, an *isotopy* or *isotopic lifting* is any lifting of it, which gives a new mathematical structure verifying the same basic axioms (or properties) as the first. This new structure is called *isotopic structure* or *isostructure* (see [2]).

In 1978 (see [2]), Santilli proposes a possible model of isotopy, called Santilli's isotopy, which allows to construct the named *mathematical isostructure*, based on an isounit I. This isounit can be obtained starting from the following definition:

Let E be any mathematical structure, defined on a set of elements C. Let $V \supseteq C$ be a set with an inner law $*$ and an unit element I. Such a set V is said to be the *general set of the isotopy*. Let $\widehat{I} \in V$ be such that its inverse $T = \widehat{I}^{-I}$, with respect to the law $*$, exists. I will be called *isotopic unit* or *isounit* and it will be the basic unit in the lifting of the structure E. T will be the *isotopic element*. Finally, $\widehat{I}$ and $*$ are the *isotopy elements*.

Then, Santilli proposes to reach an isostructure $\widehat{E}$ starting from the structure E, by considering the following construction levels:

a) **Conventional level**: (see [2]) It is the initial mathematical structure, formed by a set of elements and the laws defined among them. In this level appear the usual mathematical structures with respect to usual units: $E = E(a, +, \times, ...)$.

b) **General level**: It is the general set V, in which are, particularly, the isotopy elements used in the isoproduct construction model, that is, $V = V(\alpha, *, \star, ...)$. It is important to note that $E_* = E(a, *, \star, ...)$ (the restriction of V to E) must verify the same axioms as the initial structure E.

c) **Isotopic level**: (see [2]) It is the mathematical isostructure obtained when lifting, that is $\widehat{E} = \widehat{E}(\widehat{a}, \widehat{+}, \widehat{\times}, ...)$.

It is formed by an isotopic set and the isolaws on it. Elements of such set, which are usually denoted by using a hat, are given with respect to the isounit of $\widehat{E}$. So, fixed and given the isostructure $(\widehat{E}, \widehat{\times})$, with isounit $\widehat{I}$, where I is the unit of E with respect to $*$, these elements are $\widehat{a} = \widehat{a}\widehat{\times}\widehat{I}$, where Santilli defines the law $\widehat{\times}$ as $\widehat{a}\widehat{\times}\widehat{b} = \widehat{a * b}$. See then that, $\widehat{a}\widehat{\times}\widehat{I} = \widehat{a * I} = \widehat{a} = \widehat{I}\widehat{\times}\widehat{a}$, which implies that $\widehat{I}$ is the unit element of $\widehat{E}$ with respect to $\widehat{\times}$.

It is immediate to check that the mapping **I** $: E \to \widehat{E} : a \to \widehat{a}$ is a bijection, because it is onto by construction and it is also injective, due to $\widehat{a} \neq \widehat{b}$, for all $a, b \in E$ such that $a \neq b$. Indeed, in $\widehat{E}$, $\widehat{a} = \widehat{a}\widehat{\times}\widehat{I} \neq \widehat{b}\widehat{\times}\widehat{I} = \widehat{b}$ with respect to the isounit $\widehat{I}$ of $\widehat{E}$; in the same way as $a = a \times e \neq b \times e = b$ in E, where e is the unit element of E with respect to $\times$.

d) **Level of projection**: (see [6]) It appears when we consider the mathematical isostructure $\widehat{E}$ referred to the isotopy elements used in its construction. Its elements are denoted by a line superposed to the hat $\widehat{\ }$ of elements of $\widehat{E}$, that is, $\overline{\widehat{\ }}$.

In this way, if we use the isotopy elements $*$ (with unit I) and $\widehat{I}$ to construct $\widehat{E}$, then we obtain a structure $\overline{\widehat{E}}$ in the level of projection, whose elements are referred to the unit I: $\overline{\widehat{a}} = a * \widehat{I} = (a * \widehat{I}) * I$.

The mapping $\pi : \widehat{E} \to \overline{\widehat{E}} : \widehat{a} \to \pi(\widehat{a}) = \overline{\widehat{a}}$ is named *projection*. In general, we say that an element of $\widehat{E}$ *is projected* on its corresponding associated element belonging to $\overline{\widehat{E}}$. Note that, by construction, the mapping π is onto.

In a first stage, $\overline{\widehat{E}}$ is only doted with laws when π is linear with respect to the isolaws associated with $\widehat{E}$. So, fixed an isostructure $(\widehat{E}, \widehat{\times})$, if π is linear with respect to $\widehat{\times}$, the law $\overline{\widehat{\times}}$ is defined on $\overline{\widehat{E}}$ by $\overline{\widehat{a}}\,\overline{\widehat{\times}}\,\overline{\widehat{b}} = \overline{\widehat{a}\widehat{\times}\widehat{b}}$. In such a case, $\pi : (\widehat{E}, \widehat{\times}) \mapsto (\overline{\widehat{E}}, \overline{\widehat{\times}})$ is an onto morphism .

Therefore, this level of projection is the most important in practice, because it allows to obtain some mathematical models which would be no possible under usual units.

There exists still another level which joins both conventional and isotopic levels. It is the *axiomatic level* ([2]), which identifies every mathematical structure verifying the same axioms.

So, in a schematic way, as the different construction levels appearing in an isotopic lifting, as the relations among them can be observed in the following diagram:

Conventional level	General level $(V, *, \star, ...)$	
$(E, +, \times, ...)$	$\longrightarrow$	$(E, *, \star, ...)$
$\downarrow$	$\sharp$	$\downarrow$
Level of projection	Projection $\longleftarrow$	Isotopic level
$(\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, ...)$		$(\widehat{E}, \widehat{+}, \widehat{\times}, ...)$

Finally, we will say that an isotopic lifting of the structure E is *injective* if $X = Y$, for all $X, Y \in E$ such that $\overline{\widehat{X}} = \overline{\widehat{Y}}$. It is equivalent, by construction, to say that the projection $\pi : \widehat{E} \to \overline{\widehat{E}} : \widehat{a} \to \pi(\widehat{a}) = \overline{\widehat{a}}$ is an injective mapping. Therefore, as a consequence, if the isotopic lifting of E is injective, then $\pi : \widehat{E} \mapsto \overline{\widehat{E}}$ will be an isomorphism.

3. Isogroups

Starting from the definition of *isogroup* (see [2]), we give in this section some examples and properties of them.

Definition 1 *Let $(G, \circ)$ be a group, with an associative inner law $\circ$, and an unit element e. An* isogroup *$\widehat{G}$ is an isotopy of G, now equipped with a new inner associative law, $\widehat{\circ}$ and an unit element $\widehat{I}$, such that the pair $(\widehat{G}, \widehat{\circ})$ verifies the axioms of a group. If, besides, $\widehat{\alpha}\widehat{\circ}\widehat{\beta} = \widehat{\beta}\widehat{\circ}\widehat{\alpha}$ is verified for all $\widehat{\alpha}, \widehat{\beta} \in \widehat{G}$, then we say that $\widehat{G}$ is an* isoabelian *isogroup or* isocommutative *isogroup.*

See that this definition of isogroup is quite general. So, the unit element with respect to $\widehat{\circ}$, which we call isounit, is not, in general, the mentioned isounit in a Santilli's isotopy. However, when we do that construction, we look after to make it in this way so that these two elements were the same. Moreover, the fact of writing $\widehat{I}$ in the place of $\widehat{e}$, is not casual. If we follow the notation used just here, $\widehat{e}$ is the isotopic lifting of e, but, in general, $\widehat{e}$ is not the unit element of $\widehat{\circ}$. It implies that notions, properties and theorems studied in the initial structure cannot be applied in the new structure.

Let see it in the case that we have got a Santilli's isotopy, using a fixed isounit and a $*$ law. To do it, when we have the initial group $(G, \circ)$, we consider the isounit $\widehat{I}$ (which does not belong to G in general), and we define the law $*$ which we want to work with. It is already known the model that we use to construct the isotopic set $\widehat{G}$, by using the isounit $\widehat{I}$ and the law $*$ (which is applied for any structure). So, $\overline{\widehat{G}} = \{\overline{\widehat{\alpha}} = \alpha * \widehat{I} = \alpha\widehat{I} \mid \alpha \in G\}$.

Now, let see how to lift the associated law $\circ$. In the isotopic level, we define the new law as follows: $\widehat{\alpha}\widehat{\circ}\widehat{\beta} = \widehat{\alpha * \beta}, \forall\widehat{\alpha}, \widehat{\beta} \in \widehat{G}$. Therefore, in the level of projection, we define the new law as follows: $\overline{\widehat{\alpha}\widehat{\circ}\widehat{\beta}} = (\alpha * \beta) * \widehat{I}$. The new law is called *isoproduct*.

We can show next that if we impose that $(G, *)$ is an associative group with $I \in G$ as the unit element with respect to $*$, then $(\widehat{G}, \widehat{\circ})$ is an isogroup. To do it, we will see that $\widehat{\circ}$ is an inner law which verifies the axioms of a group.

Indeed, $\forall\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma} \in \widehat{G}$, we have:

a) $\widehat{\circ}$ is an inner law for $\widehat{G}$, since $\widehat{\alpha}\widehat{\circ}\widehat{\beta} = \widehat{\alpha * \beta} \in \widehat{G}$, because $\alpha * \beta \in G$, due to $*$ is an inner law for G by hypothesis (remember that $(G, *)$ must be a group).

b) $(\widehat{\alpha}\widehat{\circ}\widehat{\beta})\widehat{\circ}\widehat{\gamma} = \widehat{\alpha * \beta}\widehat{\circ}\widehat{\gamma} = (\widehat{\alpha * \beta) * \gamma} = \widehat{\alpha * (\beta * \gamma)} = \widehat{\alpha}\widehat{\circ}\widehat{\beta * \gamma} = \widehat{\alpha}\widehat{\circ}(\widehat{\beta}\widehat{\circ}\widehat{\gamma})$. (See that $*$ is associative is very important so that $\widehat{\circ}$ is associative).

c) $\widehat{I} \in \widehat{G}$, since $I \in G$. (See that $\overline{\widehat{I}} = I * \widehat{I}$). Moreover, $\widehat{I}$ is the isounit that we search, because $\widehat{\alpha}\widehat{\circ}\widehat{I} = \widehat{\alpha * I} = \widehat{\alpha} = \widehat{I}\widehat{\circ}\widehat{\alpha}$.

d) Let $\widehat{\alpha} \in \widehat{G}$ be. It will be $\alpha \in G$. So, as $(G, *)$ is a group with unit element I, there exists $\alpha^{-I} \in G$, such that $\alpha * \alpha^{-I} = \alpha^{-I} * \alpha = I$. Then, we must only take the element $\widehat{\alpha^{-I}}$, as the isoinverse of $\widehat{\alpha}$ with respect to $\widehat{\circ}$, because then we have that $\widehat{\alpha}\widehat{\circ}\widehat{\alpha^{-I}} = \widehat{\alpha * \alpha^{-I}} = \widehat{I} = \widehat{\alpha^{-I}}\widehat{\circ}\widehat{\alpha}$.

e) Finally, if $*$ is commutative, $(\widehat{G}, \widehat{\circ})$ will be also commutative, because $\widehat{\alpha}\widehat{\circ}\widehat{\beta} = \widehat{\alpha * \beta} = \widehat{\beta * \alpha} = \widehat{\beta}\widehat{\circ}\widehat{\alpha}$.

So, we have proved the following:

Theorem 1 *Let $(G, \circ)$ be an associative group and let $\widehat{I}$ and $*$ be two isotopy elements. If $(G, *)$ is an associative group with unit element $I \in G$, then the isotopic lifting $(\widehat{G}, \widehat{\circ})$ constructed by the model of the isoproduct, has an isostructure of isogroup. Moreover, if $(G, *)$ is commutative, then $(\widehat{G}, \widehat{\circ})$ is a commutative isogroup.* ■

Now, we will see some examples of isogroups:

Example 1 Let $(\mathbf{R}, +)$ be the group of the real numbers with the usual sum. A trivial isotopic lifting could be constructed by using the isounit $\widehat{I} = 0$ and the law $* \equiv +$ (note that $(\mathbf{R}, *) = (\mathbf{R}, +)$ is a group with unit element $0 \in \mathbf{R}$); so we would have the pair $(\widehat{\mathbf{R}}, \widehat{+})$, where $\overline{\overline{\mathbf{R}}} = \{\overline{\overline{a}} = a * 0 = a + 0 = a \mid a \in \mathbf{R}\} = \mathbf{R}$. Moreover, as $* \equiv +$, the isoproduct would be defined as $\widehat{a}\widehat{+}\widehat{b} = \widehat{a * b} = \widehat{a + b}$ and $\overline{\overline{\widehat{a}\widehat{+}\widehat{b}}} = \overline{\overline{\widehat{a + b}}} = a + b$. So, we would have $\overline{\overline{\widehat{+}}} \equiv * \equiv +$.

So, the isotopy of $(\mathbf{R}, +)$, given by the isounit 0 and the law $* \equiv +$, is the same as the trivial isotopy, that is, the identity. It proves that the construction which we are using is right, since if we do not change either the initial unit or the initial group law, then this group remains invariant when constructing the isotopy. ■

Example 2 Now, let consider the isotopy of the group $(\mathbf{R}^*, \times)$ ($\mathbf{R}^*$ is the real numbers set minus the zero), obtained by using the isounit $\widehat{I} = i$ and the law $* \equiv \bullet$ (that is, by the usual complex law).

So, the isotopic set is $\widehat{\mathbf{R}^*} = Im(\mathbf{C}) \setminus \{0\}$ and the isoproduct will be defined as $\widehat{a}\widehat{\times}\widehat{b} = \widehat{a * b} = \widehat{a \bullet b} = \widehat{a \times b}$ for all $a, b \in \mathbf{R}$. Then, $\overline{\overline{\widehat{a}\widehat{\times}\widehat{b}}} = \overline{\widehat{a}\widehat{\times}\widehat{b}} = \overline{\widehat{a \times b}} = (a \times b) * i = (a \bullet b) \bullet i$ for all $a, b \in \mathbf{R}$. ■

Moreover, let see that the isogroups of the last two examples are isocommutative. It can be observed by using Theorem 1, because we have that the initial groups are commutative.

Note that in both examples it has been used an isounit acting as a constant. However, examples in which the isounit used depends on initial coordinates can be also shown:

Example 3 Let consider $(\mathbf{R}, +)$, as in Example 1. We consider an isotopic lifting with isotopy elements $* \equiv +$ and $\widehat{I} = \widehat{I}(x) = \left\{ \begin{array}{c} 1, \text{ if } x = 0 \\ \frac{1}{x^2}, \text{ if } x \neq 0 \end{array} \right\}$. Then, $\widehat{I}$ is positive defined and non-singular and thus we obtain the lifting $a \to \overline{\overline{a}} = a * \widehat{I} = \left\{ \begin{array}{c} 0, \text{ if } a = 0 \\ \frac{1}{a}, \text{ if } a \neq 0 \end{array} \right\}$.

Finally, the isoproduct is defined by $\widehat{a}\widehat{+}\widehat{b} = \widehat{a + b}$, where $\overline{\overline{\widehat{a}\widehat{+}\widehat{b}}} = \left\{ \begin{array}{c} 0, \text{ if } a + b = 0 \\ \frac{1}{a+b}, \text{ if } a + b \neq 0 \end{array} \right\}$ in the projection level. In this way, the set $\{0\} \cup \{\frac{1}{a} : a \in \mathbf{R}^*\}$ can be doted of an isogroup structure, by the law $\overline{\overline{\widehat{+}}}$. ■

To finish this section we will prove that fixed and given an arbitrary isogroup, it can be considered as an isotopic lifting which follows the isotopic construction model which we are considering. Indeed, we have the following:

Proposition 1 *Every isotopy* $\mathbf{I} : (G, \circ) \to (\widehat{G}, \widehat{\circ})$ *can be considered as an isotopic lifting which follows the isoproduct construction model.*

PROOF. It is sufficient to consider in the general level the set $(G, *)$, where $*$ is defined by $a * b = \mathbf{I}^{-1}(\widehat{a}\widehat{\circ}\widehat{b})$ (it has sense, because the mapping $\mathbf{I} : E \to \widehat{E} : a \to \widehat{a}$ is a bijection by construction, as we already observed). So, the isolaw $\widehat{\circ}$ can be defined later as $\widehat{a}\widehat{\circ}\widehat{b} = \mathbf{I}(a * b) = \widehat{a * b}$, and this is the way in which an isolaw is defined according to the isoproduct construction level, as we already saw.

In this way, we also get, by linearity, to dote the set $(G, *)$ with a structure of group. Moreover, the unit corresponding to $*$ will be, by construction, the element of E, from which the isounit of the associated isolaw $\widehat{\circ}$ is isotopically lifted. ■

4. Isosubgroups

In this section we introduce the definition of *isosubgroup*. We give some examples and properties of them and we also deal with the differences between an isosubgroup and a subgroup of an isogroup.

We study next possible liftings of the subgroups, starting from the Santilli's construction model. To do this, we must demand that an isosubgroup is the isotopic lifting of a subgroup H of a given and fixed group G. The difficulty appears when we require that every isotopic lifting of a given structure is also a structure of the same type. So, every isotopy of H should have structure of subgroup and thus, the isotopic liftings of H could not be independent of the lifting of G. Therefore, the definition of isosubgroup would be as follows:

Definition 2 *Let* $(G, \circ)$ *be an associative group and* $(\widehat{G}, \widehat{\circ})$ *be an associated isogroup. Let* H *be a subgroup of* G*. We say that* $\widehat{H}$ *is an* isosubgroup *of* $\widehat{G}$ *if, being an isotopy of* H*, the pair* $(\widehat{H}, \widehat{\circ})$ *is a subgroup of* $\widehat{G}$*, that is, if* $\widehat{H} \subseteq \widehat{G}$*,* $\widehat{\circ}$ *is an inner law in* $\widehat{H}$ *and* $(\widehat{H}, \widehat{\circ})$ *has structure of group.*

Let apply now this definition to the Santilli's construction model, by an isounit and a law $*$. Suppose that we have the associative group $(G, \circ)$ and the isogroup $(\widehat{G}, \widehat{\circ})$, obtained by both, an isounit $\widehat{I}$ and a law $*$ previously fixed. Let H be a subgroup of G. Since we demand that in the future isosubgroup $\widehat{H}$, the associated law is $\widehat{\circ}$, if we go on about the given construction of the isoproduct, then the law and the isounit (both which will be the isotopy elements), must be, respectively, $*$ and $\widehat{I}$, because if not, we would not obtain the same law $\widehat{\circ}$ in general. See it with an example:

Example 4 Let $(\mathbf{Z}, +)$ be the group of integers, with the usual sum. We take, under usual notations, $* \equiv +$, $\widehat{I} = 2$. As $(\mathbf{Z}, *) = (\mathbf{Z}, +)$ is a group with unit element $0 \in \mathbf{Z}$, we can use the isotopy of elements $*$ and $\widehat{I}$. Then, $\overline{\widehat{\mathbf{Z}}} = \{\overline{\widehat{a}} = a * 2 = a + 2 \mid a \in \mathbf{Z}\} = \mathbf{Z}$ and the isoproduct is defined as $\widehat{a}\widehat{+}\widehat{b} = \widehat{a + b}$, being $\overline{\widehat{a}}\overline{\widehat{+}}\overline{\widehat{b}} = \overline{\widehat{a + b}} = (a + b) * 2 = a + b + 2$, for all $a, b \in \mathbf{Z}$. In this way, we have obtained the isogroup $(\widehat{\mathbf{Z}}, \widehat{+})$, which comes from the additive group $(\mathbf{Z}, +)$.

Let now consider the subgroup $(\mathbf{P}, +)$ of even integers and 0. If we construct the isotopy related to the same elements as above (which is always possible, due to $(\mathbf{P}, *) = (\mathbf{P}, +)$ is a group with unit element $0 \in \mathbf{P}$), we obtain firstly the isotopic set $\widehat{\mathbf{P}}$, being $\overline{\widehat{\mathbf{P}}} = \{\overline{\widehat{m}} = m * 2 = m + 2 \mid m \in \mathbf{P}\} = \mathbf{P}$, and we would get after the same isoproduct $\widehat{+}$.

Let now prove that $(\widehat{\mathbf{P}}, \widehat{+})$ is a isosubgroup of $(\widehat{\mathbf{Z}}, \widehat{+})$, taking into consideration that $\widehat{\mathbf{P}}$ is an isotopy of $\mathbf{P} \subseteq \mathbf{Z}$. To see it, we observe

a) $\widehat{\mathbf{P}} \subseteq \widehat{\mathbf{Z}}$, since $\mathbf{P} \subseteq \mathbf{Z}$.

b) $\widehat{m}\widehat{+}\widehat{n} = \widehat{m+n} \in \widehat{\mathbf{P}}$, for all $m, n \in \mathbf{P}$. So, $\widehat{+}$ is an inner law on $\widehat{\mathbf{P}}$.

c) $\widehat{P}$ satisfies the group conditions, because associativity is from $(\widehat{G}, \widehat{\circ})$, the isounit $\widehat{I} = 2 = \widehat{0}$ belongs to $\widehat{\mathbf{P}}$ and $\widehat{m}^{-\widehat{I}} = \widehat{-m} \in \widehat{\mathbf{P}}$, $\forall m \in \mathbf{P}$, since $\widehat{m}\widehat{+}\widehat{-m} = \widehat{m+(-m)} = \widehat{0} = \widehat{I} = \widehat{-m}\widehat{+}\widehat{m}$.

So, $(\widehat{\mathbf{P}}, \widehat{+})$ is an isosubgroup of $(\widehat{\mathbf{Z}}, \widehat{+})$. ■

Note that in this example, we can avoid some steps when constructing $\widehat{H}$, once it is proved that we can make the isotopy corresponding to elements used to construct $\widehat{G}$. Indeed, remember that by using Theorem 1, conditions to be satisfied are that the pair $(H, *)$ is a group, having the unit element I, the same of $(G, *)$. So, if we proceed similarly as we did when proving that $(\widehat{G}, \widehat{\circ})$ had a group structure by the isoproduct $\widehat{\circ}$ (obtained starting of $*$), we have the conditions needed: $\widehat{+}$ is an inner law on $\widehat{\mathbf{P}}$, the group axioms are satisfied by construction and finally, $(\widehat{H}, \widehat{\circ})$ is associative due to $\widehat{\circ}$ is associative on $(\widehat{G}, \widehat{\circ})$, for $*$ being it by hypothesis.

Therefore, it is proved the following:

Theorem 2 *Let $(G, \circ)$ be an associative group and $(\widehat{G}, \widehat{\circ})$ be the associated isogroup corresponding to the isotopy of elements $\widehat{I}$ and $*$. Let H be a subgroup of G. If $(H, *)$ has structure of subgroup of $(G, *)$, then the isotopic lifting $(\widehat{H}, \widehat{\circ})$, corresponding to the isotopy of elements $\widehat{I}$ and $*$, is a isosubgroup of $\widehat{G}$.* ■

In fact, as the isogroup construction model already pointed out this condition, we can say that if the isotopy corresponding to $\widehat{I}$ and $*$ can be made, then $(\widehat{H}, \widehat{\circ})$ is an isosubgroup of $\widehat{G}$. So, the last problem which could appear is that such an isotopy could not be made for not verifying some initial conditions. We will see it in the following example:

Example 5 Let consider the group $(\mathbf{Z}/\mathbf{Z_2}, +)$ with the usual law $+$. Let write $\mathbf{1} = 1 + \mathbf{Z}/\mathbf{Z_2}$ and $\mathbf{2} = 2 + \mathbf{Z}/\mathbf{Z_2}$. Consider now $\widehat{I} = \mathbf{1}$ and the law $*$ defined by

$$\mathbf{1} * \mathbf{1} = \mathbf{1} = \mathbf{0} * \mathbf{0}$$

$$\mathbf{1} * \mathbf{0} = \mathbf{0} * \mathbf{1} = \mathbf{0}.$$

It is easy to see that $*$ is associative, due to:

$(\mathbf{1} * \mathbf{1}) * \mathbf{1} = \mathbf{1} * \mathbf{1} = \mathbf{1} * (\mathbf{1} * \mathbf{1})$
$(\mathbf{1} * \mathbf{1}) * \mathbf{0} = \mathbf{1} * \mathbf{0} = \mathbf{1} * (\mathbf{1} * \mathbf{0})$
$(\mathbf{1} * \mathbf{0}) * \mathbf{0} = \mathbf{0} * \mathbf{0} = \mathbf{1} = \mathbf{1} * \mathbf{1} = \mathbf{1} * (\mathbf{0} * \mathbf{0})$
$(\mathbf{0} * \mathbf{0}) * \mathbf{0} = \mathbf{1} * \mathbf{0} = \mathbf{0} = \mathbf{0} * \mathbf{1} = \mathbf{0} * (\mathbf{0} * \mathbf{0})$

whereas other possible cases are also satisfied by commutativity. Therefore, $(\mathbf{Z}/\mathbf{Z}_2, *)$ has structure of group, with unit element $I = \widehat{I} = \mathbf{1} \in \mathbf{Z}/\mathbf{Z_2}$.

Moreover, if we make now the isotopy corresponding to $\widehat{I}$ and $*$, we obtain that the isotopic set is $\widehat{\mathbf{Z}/\mathbf{Z_2}}$, being $\overline{\widehat{\mathbf{Z}/\mathbf{Z_2}}} = \{\overline{\widehat{\mathbf{0}}} = \mathbf{0}, \overline{\widehat{\mathbf{1}}} = \mathbf{1}\} = \mathbf{Z}/\mathbf{Z_2}$.

Besides, the corresponding isoproduct $\widehat{+}$ will be given by

$$\widehat{\mathbf{0}}\widehat{+}\widehat{\mathbf{0}} = \widehat{\mathbf{0} * \mathbf{0}} = \widehat{\mathbf{1}}$$

$$\widehat{\mathbf{0}}\widehat{+}\widehat{\mathbf{1}} = \widehat{\mathbf{0} * \mathbf{1}} = \widehat{\mathbf{0}} = \widehat{\mathbf{1}}\widehat{+}\widehat{\mathbf{0}}$$

$$\widehat{\mathbf{1}}\widehat{+}\widehat{\mathbf{1}} = \widehat{\mathbf{1} * \mathbf{1}} = \widehat{\mathbf{1}}$$

So, we have:

$$\mathbf{0}\overline{\overline{\widehat{+}}}\mathbf{0} = \overline{\overline{\widehat{\mathbf{0}}\widehat{+}\widehat{\mathbf{0}}}} = \overline{\overline{\widehat{\mathbf{1}}}} = \mathbf{1}$$

$$\mathbf{0}\overline{\overline{\widehat{+}}}\mathbf{1} = \overline{\overline{\widehat{\mathbf{0}}\widehat{+}\widehat{\mathbf{1}}}} = \overline{\overline{\widehat{\mathbf{0}}}} = \mathbf{0} = \mathbf{1}\overline{\overline{\widehat{+}}}\mathbf{0}$$

$$\mathbf{1}\overline{\overline{\widehat{+}}}\mathbf{1} = \overline{\overline{\widehat{\mathbf{1}}\widehat{+}\widehat{\mathbf{1}}}} = \overline{\overline{\widehat{\mathbf{1}}}} = \mathbf{1}$$

Thus, $\overline{\overline{\widehat{+}}} \equiv *$ and so, $(\widehat{\mathbf{Z}/\mathbf{Z_2}}, \widehat{+})$ is a new isogroup, being $(\overline{\overline{\widehat{\mathbf{Z}/\mathbf{Z_2}}}}, \overline{\overline{\widehat{+}}}) = (\mathbf{Z}/\mathbf{Z_2}, *)$.

Let consider separately the subgroup $(\{\mathbf{0}\}, +)$ of $(\mathbf{Z}/\mathbf{Z_2}, +)$. We observe that $(\{\mathbf{0}\}, *)$ does not have a structure of group, because $*$ is not an inner law on $\{\mathbf{0}\}$, due to $\mathbf{0} * \mathbf{0} = \mathbf{1} \notin \{\mathbf{0}\}$. So, conditions of Theorem 2 to obtain an isosubgroup by making the isotopy of element $\widehat{I} = \mathbf{1}$ and $*$, are not satisfied. We would have, in fact, that $\widehat{\{\mathbf{0}\}} = \{\widehat{\mathbf{0}}\}$, where $\widehat{\mathbf{0}}\widehat{+}\widehat{\mathbf{0}} = \widehat{\mathbf{1}} \notin \widehat{\{\mathbf{0}\}}$. ■

We are going to ask ourselves a new question. Suppose that we have an associative group $(G, \circ)$, with unit element I, and let $(\widehat{G}, \widehat{\circ})$ be the isogroup associated to the isotopy of elements $\widehat{I}$ and $*$. We know that every isosubgroup of $\widehat{G}$ has structure of subgroup. We also know some examples in which subgroups of G do not give rise to isosubgroup of $\widehat{G}$, by using the same $\widehat{I}$ and $*$ like isotopy elements. Then, we can finally ask if every subgroup of $\widehat{G}$ has a structure of isosubgroup of $(\widehat{G}, \widehat{\circ})$, that is, if it comes from the isotopic lifting of a subgroup of G. Of course, we have already noted that if we fixe a subgroup $(\widehat{H}, \widehat{\circ})$ of $(\widehat{G}, \widehat{\circ})$, then the law $\widehat{\circ}$ has to be the same in both pairs, so the corresponding elements of isotopic have to coincide. That is, if $\widehat{H}$ is an isosubgroup, then it have to come from an isotopy having the same elements as the ones used in the construction of $\widehat{G}$. So, the only possible subset of G which would give rise to the possible isosubgroup would be $H = \{a \in G : \widehat{a} \in \widehat{H}\} \subseteq G$. However, the pair $(H, \circ)$ is not a subgroup of $(G, \circ)$ in general, as we can check in the following:

Example 6 Let consider both the group $(\mathbf{Z}/\mathbf{Z_2}, +)$ and the isogroup $(\widehat{\mathbf{Z}/\mathbf{Z_2}}, \widehat{+})$ mentioned in the last example. We have the pairs $(\{\mathbf{0}\}, +)$ and $(\{\widehat{\mathbf{1}}\}, \widehat{+})$ respectively, as the only proper subgroups of both.

As we have just seen, if we take the subgroup $\widehat{H} = (\{\widehat{\mathbf{1}}\}, \widehat{+})$ of $(\widehat{\mathbf{Z}/\mathbf{Z_2}}, \widehat{+})$, the only possible subset of $\mathbf{Z}/\mathbf{Z_2}$ from which we could give a structure of isogroup to $\widehat{H}$ would be $H = \{\mathbf{1}\}$, because $\mathbf{1} * \mathbf{1} = \mathbf{1}$, where $\widehat{I} = \mathbf{1}$ is the isounit used in that example to construct the isotopy. However, $(\{\mathbf{1}\}, +)$ is not a subgroup of $(\mathbf{Z}/\mathbf{Z_2}, +)$, because, for instance, $+$ is not an inner law on $\{\mathbf{1}\}$, due to $\mathbf{1} + \mathbf{1} = \mathbf{0}$. ■

So, with this example, the last question above mentioned is answered in the negative. In this way, the link between groups and isogroups is finally solved.

5. Isorepresentation of finite isogroups

In this section we try to have some relations between Santilli's isotheory and the standard theory of representation of groups.

Let us recall that a representation of a finite group G is a pair (V, ρ), where V is a vector k-space and ρ is a group homomorphism $\rho : G \to Gl(V)$.

To generalize this concept on both isotopic and projection levels we firstly give the definition of *isohomomorphism of isogroups* and secondly, of *isorepresentation of finite isogroups*.

Definition 3 *Let $(G, \circ)$ and $(G', \bullet)$ be two groups and let $(\widehat{G}, \widehat{\circ})$ and $(\widehat{G'}, \widehat{\bullet})$ be associated isogroups, respectively. An* isohomomorphism *defined between $\widehat{G}$ and $\widehat{G'}$ is the isotopic lifting of any mapping $\rho : G \to G'$, that is, $\widehat{\rho} : \widehat{G} \to \widehat{G'} : \widehat{g} \to \widehat{\rho}(\widehat{g}) = \widehat{\rho(g)}$, verifying $\widehat{\rho}(\widehat{g}\widehat{\circ}\widehat{h}) = \widehat{\rho}(\widehat{g})\widehat{\bullet}\widehat{\rho}(\widehat{h})$, for all $\bigoplus \widehat{g}, \widehat{h} \in \widehat{G}$.*

Note that by demanding the compatibility of the lifting used, we obtain that ρ is a groups homomorphism with respect to the laws of G and G'. To see it we firstly introduce the following result, which is easy to prove:

Proposition 2 *Let $(G, \circ)$ be a group and $(\widehat{G}, \widehat{\circ})$ an associated isogroup. If $\widehat{G}$ has been obtained starting from an isotopy compatible with respect to $\circ$ (that is, $\widehat{g}\widehat{\circ}\widehat{h} = \widehat{g \circ h}$, for all $g, h \in G$), then G and $\widehat{G}$ are isomorphic groups.* $\square$

Moreover, if we consider the isoproduct construction model (which is always possible), we deduce the following:

Corollary 1 *Under the hypothesis of Proposition 2, if the isotopy used follows the isoproduct construction model, then we have, in the general level of the group $(G, *)$, that $(G, *) \equiv (G, \circ)$.*

PROOF.

It is immediate by construction, because fixed and given $a, b \in G$, we have $\widehat{a * b} = \widehat{a}\widehat{\circ}\widehat{b} = \widehat{a \circ b}$. So, $a * b = a \circ b$. $\blacksquare$

It is now possible to prove the following:

Proposition 3 *Under conditions of Definition 3, if the isotopy used to construct $\widehat{G}$ and $\widehat{G'}$ is compatible with respect to $\circ$ and $\bullet$, respectively, then $\widehat{\rho}$ is an isohomomorphism between $\widehat{G}$ and $\widehat{G'}$ if and only if ρ is an homomorphism between G and G'.*

PROOF.

a) $\Rightarrow$
Let us suppose that $\widehat{\rho} : \widehat{G} \mapsto \widehat{G'}$ is an isohomomorphism.

Then, fixed $g, h \in G$, we have $\widehat{\rho(g \circ h)} = \widehat{\rho}(\widehat{g \circ h}) = \widehat{\rho}(\widehat{g}\widehat{\circ}\widehat{h}) = \widehat{\rho}(\widehat{g})\widehat{\bullet}\widehat{\rho}(\widehat{h}) = \widehat{\rho(g) \bullet \rho(h)}$, and thus, $\rho(g \circ h) = \rho(g) \bullet \rho(h)$, which implies that ρ is an homomorphism, due to g and h are arbitrary in G.

b) $\Leftarrow$
Let us suppose that $\rho : G \mapsto G'$ is an homomorphism. Fixed g and h in G, we have that $\widehat{\rho}(\widehat{g}\widehat{\circ}\widehat{h}) = \widehat{\rho}(\widehat{g \circ h}) = \widehat{\rho(g \circ h)} = \widehat{\rho(g) \bullet \rho(h)} = \widehat{\rho}(\widehat{g})\widehat{\bullet}\widehat{\rho}(\widehat{h})$. So, $\widehat{\rho}$ is an isohomomorphism. $\blacksquare$

Finally, we are going to deal with the concept of isohomomorphism in the level of projection.

Definition 4 *Under conditions of Definition 3, if the isotopy used to construct $\widehat{G}$ and $\widehat{G'}$ is injective, we say that $\overline{\widehat{\rho}} : \overline{\widehat{G}} \to \overline{\widehat{G'}} : \overline{\widehat{g}} \to \overline{\widehat{\rho}}(\overline{\widehat{g}}) = \overline{\widehat{\rho}(\widehat{g})} = \overline{\widehat{\rho(g)}}$ is an* isohomomorphism *between* $\overline{\widehat{G}}$ and $\overline{\widehat{G'}}$.

Note that fixed $\overline{\widehat{g}}$ and $\overline{\widehat{h}}$ en $\overline{\widehat{G}}$, we have that $\overline{\widehat{\rho}}(\overline{\widehat{g}}\,\overline{\widehat{\circ}}\,\overline{\widehat{h}}) = \overline{\widehat{\rho}}(\overline{\widehat{g}\widehat{\circ}\widehat{h}}) = \overline{\widehat{\rho}(\widehat{g}\widehat{\circ}\widehat{h})} = \overline{\widehat{\rho}(\widehat{g})\widehat{\bullet}\widehat{\rho}(\widehat{h})} = \overline{\widehat{\rho}}(\overline{\widehat{g}})\overline{\widehat{\bullet}}\,\overline{\widehat{\rho}}(\overline{\widehat{h}})$. It shows that $\overline{\widehat{\rho}}$ is indeed a groups homomorphism. Moreover, by construction, the following result is reached:

Proposition 4 *Let G and G' be two groups and let $\widehat{G}$ and $\widehat{G'}$ be the corresponding associated isogroups. Then, we have:*

a) If the isotopic lifting used to construct $\widehat{G}$ and $\widehat{G'}$ is injective, then $\overline{\widehat{\rho}} : \overline{\widehat{G}} \mapsto \overline{\widehat{G'}}$ is an isohomomorphism if and only if $\widehat{\rho} : \widehat{G} \mapsto \widehat{G'}$ is an isohomomorphism.

b) If such a lifting is also compatible with respect to the laws on G and G', then the following assertions are equivalent:

b.1) ρ is a groups homomorphism in the conventional level.

b.2) $\widehat{\rho}$ is an isogroups isohomomorphism in the isotopic level.

b.3) $\overline{\widehat{\rho}}$ is an isogroups isohomomorphism in the level of projection. ■

Now, we can already introduce the definition of isorepresentation of a finite isogroup:

Definition 5 *Let G be a finite group, k a field and V a vector k-space. Let $\widehat{G}, \widehat{k}$ and $\widehat{V}$ be isotopic liftings of the previous structures. An* isorepresentation *of $\widehat{G}$ is a pair $(\widehat{V}, \widehat{\rho})$, where $\widehat{\rho} : \widehat{G} \to \widehat{GL}(\widehat{V})$ is an isogroups isohomomorphism, with $\widehat{GL}(\widehat{V}) = \widehat{GL(V)} = \left\{\widehat{f} : \widehat{V} \to \widehat{V} : \widehat{v} \to \widehat{f}(\widehat{v}) = \widehat{f(v)}\right\}$.*

If the isotopic lifting used to construct $\widehat{G}$ and $\widehat{GL}(\widehat{V})$ is injective, an isorepresentation *of $\overline{\widehat{G}}$ is a pair $(\overline{\widehat{V}}, \overline{\widehat{\rho}})$, where $\overline{\widehat{\rho}} : \overline{\widehat{G}} \to \overline{\widehat{GL}}(\overline{\widehat{V}})$ is an isogroups isohomomorphism in the level of projection, with $\overline{\widehat{GL}}(\overline{\widehat{V}}) = \overline{\widehat{GL}(\widehat{V})}$.*

As a consequence of Proposition 4, it is immediate the following:

Corollary 2 *Under conditions of Definition 5, we have:*

a) If the isotopic lifting used to construct $\widehat{G}$ and $\widehat{GL}(\widehat{V})$ is compatible with respect to the laws on G and $GL(V)$, then $(\widehat{U}, \widehat{\rho})$ is an isorepresentation of $\widehat{G}$ if and only if (V, ρ) is a representation of G.

b) If such a lifting is injective, then $(\overline{\widehat{V}}, \overline{\widehat{\rho}})$ is a isorepresentation of $\overline{\widehat{G}}$ if and only if $(\widehat{V}, \widehat{\rho})$ is an isorepresentation of $\widehat{G}$. ■

To finish the paper, we also try in the next section to set some links between this isotheory and the standard groups theory, referred to equivalence relations defined on groups.

6. Isotopically equivalent groups

In general, fixed and given an algebraic structure E, an isostructure associated $\widehat{E}$ is characterized by verifying the same axioms as E. In this section, we give some results obtained when studying relations between E and $\widehat{E}$. Particularly, we deal with the case of E being a group. We introduce the following:

Definition 6 *A group $(G, \circ)$ is* isotopically related *with another group $(G', \bullet)$ if $(G, \circ)$ is the projection of an isogroup $(\widehat{G'}, \widehat{\bullet})$ associated with $(G', \bullet)$, that is, if $(G, \circ) \equiv (\overline{\widehat{G'}}, \overline{\widehat{\bullet}})$.*

It is easy to check that to demand the previous relation to be of equivalence, the projection mentioned should be a groups isomorphism. So, we give another definition, which particularizes the previous one:

Definition 7 *A group $(G, \circ)$ is* isotopically equivalent *to another group $(G', \bullet)$ (it will be denoted by $(G, \circ) \sim (G', \bullet)$), if it exists an isogroup $(\widehat{G'}, \widehat{\bullet})$ associated with $(G', \bullet)$, such that:*

a) $(G, \circ) \equiv (\overline{\widehat{G'}}, \overline{\widehat{\bullet}})$.

b) $(G, \circ) \cong (\widehat{G'}, \widehat{\bullet})$.

Note that as we always can use the isoproduct construction model to lift structures, the following result can be proved:

Proposition 5 *A group $(G, \circ)$ is isotopically equivalent to another group $(G', \bullet)$ if and only if a law $*$ can be defined on G', in such a way that $(G, \circ)$ and $(G', *)$ are isomorphic.* $\square$

Note that this relation defined on groups, consisting on *be isotopically equivalent*, that is, $(G, \circ) \sim (G', \bullet)$, does not implies, in general, that $(G, \circ) \cong (G', \bullet)$. However, it is easily verifiable that this relation between groups is reflexive, symmetric and transitive and thus, it is an equivalence relation. It generalized the relation of groups isomorphism.

Moreover, if we reduce this study to the particular case of finite groups, it can be proved the following:

Corollary 3 *Two finite groups $(G, \circ)$ and $(G', \bullet)$ are isotopically equivalent if and only if $|G| = |G'|$, where $|G|$ denotes the cardinal of the group G.* $\square$

To finish this section, we are going to deal briefly with an application of the previous concepts. We will consider the symmetry groups of a planar figure.

When we say *planar figure* we mean any subset of points belonging to Euclidean Plane $\pi_{X,Y}$. Every planar figure F is associated with its symmetry group, denoted by S_F. Note, however, that S_F can be the symmetry group of more than one planar figure, in spite of S_F being unique with respect to F. Therefore, planar figures can be classified according to the symmetry group.

Another way to classify planar figures is by considering the relation of isotopically equivalence among symmetry groups. To do this, we introduce the following:

Definition 8 *Two planar figures F and F' are* isotopically symmetric *(or F and F' has isotopically equivalent symmetries), if their respective symmetry groups S_F and $S_{F'}$ are isotopically equivalent.*

The following result is an immediate consequence of the previous definition and the equivalence relation of isotopically equivalent groups:

Corollary 4 *The relation* be isotopically symmetric *defined on the set of planar figures is of equivalence.* ■

Acknowledgement. Authors gratefully acknowledge Professor R. M. Santilli for the useful help given.

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